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Cubic surfaces and their invariants: Some memories of Raymond Stora

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Abstract

Cubic surfaces embedded in complex projective 3-space are a classical illustration of the use of old and new methods in algebraic geometry. Recently, they made their appearance in physics, and in particular aroused the interest of Raymond Stora, to the memory of whom these notes are dedicated, and to whom I'm very much indebted.

Each smooth cubic surface has a rich geometric structure, which I review briefly, with emphasis on the 27 lines and the combinatorics of their intersections. Only elementary methods are used, relying on first order perturbation/deformation theory.

I then turn to the study of the family of cubic surfaces. They depend on 20 parameters, and the action of the 15 parameter group $SL_4(\mathbb{C})$ splits the family in orbits depending on 5 parameters. This takes us into the realm of (geometric) invariant theory. I review briefly the classical theorems on the structure of the ring of polynomial invariants and illustrate its many facets by looking at a simple example, before turning to the already involved case of cubic surfaces. The invariant ring was described in the 19th century. I show how to retrieve this description via counting/generating functions and character formulae.

Theses notes are dedicated to Raymond Stora. As many a student in theoretical physics in the 1980's, I had heard about him when I learned the basics of the quantization of gauge theories. My first opportunity to meet him in real life was in Annecy in March 1990 on the occasion of his

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60th birthday, but I have no memory of a personal contact on this occasion. At the end of the same year, Raymond came to Saclay as a member of the jury for my PhD thesis, and on that occasion we had our first real discussion – something related to line bundles over the complex projective line if I remember correctly – which Raymond completed by sending me a letter, the first of a long series. We met on a few other occasions before I arrived as a fellow at CERN in 1993. During the two years I spent there, we had almost daily thorough conversations in Raymond's office and at the cafeteria. Though no publication came out, it was a period when I learned a lot about things that would prove useful for me later on. After I left CERN to return to Saclay, I spent a few weeks in Annecy each year for a few years and the discussions with Raymond went on. During the decade that followed, we had very few occasions to see each other. However, I called Raymond a few times each year. The typical scenario was the following: I had come across a question and it reminded me vaguely about an old conversation. I had forgotten most or all the details, so Raymond had to tell the story all over, sometimes even with a few bonuses. When I hang down the phone, I had either the solution to my problem or at least a much clearer and broader picture. These phone calls were often completed by letters from Raymond. The end of the phone calls or of the letters were devoted to Raymond's problems, which always raised my interest, though I could most of the time contribute only very modestly, if at all, to the solution. When Raymond's colleagues from Annecy decided to celebrate (one year late) Raymond's 80th birthday in 2011, they kindly asked me to be a member of the organizing committee. I accepted with pleasure. As it turned out, I did nothing concrete during the preparation, so I decided that at least I should show up, and I organized to spend my vacations near Annecy that summer. I've done the same ever since, and this gave me the opportunity to visit Raymond each year once or twice at CERN for a lunch and an afternoon chat. The lunch was also an opportunity to meet my dear friends Anne-Marie, Marie-Noelle, and Suzy, who have contributed so much to make my life as a fellow at CERN-TH enjoyable by their permanent good mood, receptiveness, dynamism, dedication and kindness.

My father in Physics, i.e. my thesis adviser, was the late Claude Itzykson, and in that sense Raymond Stora was my grandfather.[1] Both Claude and Raymond were very fond of elementary geometry. Raymond was educated in the French system around the middle of the 20th century. At the time, classical geometry was a very fashionable subject at school. I think it is fair to say that Raymond's approach to problems in physics was very often grounded in geometry. Over the last twenty five years I had a number of opportunities to hear Raymond make the statement that he was doing some trigonometry to help his colleagues. In fact he was helping them to attack difficult problems, and I think his use of the word "trigonometry" was a reflection of Raymond's modesty: he probably thought that trigonometry sounded more elementary and less pretentious than geometry. But what he was really doing was geometry.

Claude and Raymond also had a serious background in group theory.[2] However, though Claude made constant use of representation theory for finite groups and for Lie groups throughout his career, Raymond was more on the side of the study of group actions (gauge theories, and gauge fixing in particular, being a typical example). Anyway, at their contact, I had no choice but to love group theory.

[1] Raymond and my father were born on the same year. Both Claude and Raymond behaved very much like parents with me and I remember that, despite honest and serious efforts, I never managed to pay my share, not to speak about paying something for them, when I had a drink or a meal with one of them.

[2] Of course, group theory is a close relative of geometry, as explained in Klein's Erlanger program.

Raymond knew it, and for my good fortune he got in touch with me in the fall of 2014 to discuss about invariants of (the defining equations of) cubic surfaces.[3] This is the motivation for the following exposition.

1. Introduction

The question raised by Raymond has its roots in his last published paper [6] to which the reader is referred to get more background. Let me simply say that homogeneous cubic polynomials play a central role in supersymmetric Born–Infeld Lagrangians.

A homogeneous cubic polynomial is essentially the same thing as a symmetric 3-tensor via the correspondence:

$$P(x_1, \cdots, x_{n+1}) = \frac{1}{3} \sum_{a,b,c=1}^{n+1} d_{abc} x_a x_b x_c.$$

The critical points are the zeros of the partial derivatives

$$\frac{\partial P}{\partial x_c} = 0 \text{ i.e. } \sum_{a,b}^{n+1} d_{abc} x_a x_b = 0.$$

Oversimplifying grossly, this is a collection of quadratic equations that have to be solved to get to the physics. The structure of the solution set depends on the coefficients d_{abc}, but linear transformations of the x_as have a predictable effect and split the space of symmetric tensors into orbits. A manageable (or even better a canonical) representative of the d_{abc}s in each orbit is the right basis to understand the set of solutions. A crucial tool is to list the polynomials invariants, i.e. the polynomial in the d_{abc}s that are constant on the orbits.

The case of homogeneous cubic polynomials in three variables is closely related to the theory of elliptic curves (see e.g. [14,17]) and is treated in detail in [6]. One of the subtleties is that physics applications require the study of the *real* case. However, the result shows that a thorough understanding of the *complex* situation is a prerequisite and already gives a reasonable picture.

Raymond had decided to look at the next case in order of difficulty, cubic polynomials in four variables and their invariants. As he quickly realized, the structure of the invariant ring and the complexity of individual invariants, plus the richness of possible degeneration of cubic surfaces and the additional difficulties with the real case, suggests that the road to a complete understanding of the physics in that case would be a formidable task.

But at that time Raymond was forced to spend a macroscopic fraction of his time at the hospital and he needed a good way to keep his neurons busy. He was not totally satisfied with the available literature. The fundamental references on invariant theory, specifically [19,20] for cubic surfaces for instance, are old and difficult to read nowadays. The vocabulary invented then to describe specific constructions of invariants is rich but unfamiliar today, and it is not so obvious to grasp the line of reasoning and the exact meaning of the results. Nevertheless, the theory accumulated by the classical geometers of the 19th century is really impressive. The more modern references on cubic surfaces and invariants, for instance [5,4] require (or cover first) some serious background in algebraic geometry and lack detailed down to earth computations. Both ancient

[3] The following discussion is informal, but precise definitions and statement will be given in the main text.

and modern works state that the ring of invariants is generated by five independent homogeneous polynomials of degree 8, 16, 24, 32, 40 and a (so-called skew invariant) polynomial of degree 100 whose square is a polynomial in the previous five.

After some direct attempts to understand the situation, we decided with Raymond that as a first step, we should at least count the number of homogeneous invariant polynomials of a given degree, a task which is in principle purely mechanical, relying on the Weyl character formula for irreducible representations of the special linear group. I had old notes on this approach, dating back to my years as a fellow at CERN in the 1990's in fact. I sent those to Raymond. Though purely mechanical, we soon realized that even this modest part of the problem would not come for free: my previous experience of explicit computations was on much simpler examples (simpler representations of simpler groups) and already relied on the computer. The case at hand seemed to require much more memory and a lot of craftsmanship to help the machine. Knowing Raymond and the extreme care he took to have nothing to do with computers whatsoever gives a special taste of irony to the story. Raymond read in detail [2,23] and found a few interesting improvements that turned out to make the difference in the end. It is a pity that I could only carry out the last steps after Raymond died on July 20, 2015.

These notes are a *promenade* through the geometry of *complex* cubic polynomials in four variables and their invariants under the action of the *complex* special linear group. There is little or no claim at originality. But there is some hope that this down-to-earth presentation complements the more abstract approaches.

Section 3 is closely related to the content of seven long letters from Raymond, dated from September 30, 2014 to May 22, 2015 and of the weekly discussions we had with Raymond over the phone during the same period. I have tried to give enough background to make the discussion mostly self-contained.

- After the basic definitions, I quote a structure theorem for the ring of polynomial invariants of finite dimensional representations of $SL_n(\mathbb{C})$. I use it to give the general features of the Molien series, which count the dimension of homogeneous invariant polynomials in each degree. I also make a few obvious remarks on the separation of orbits by invariants. I barely scratch the surface of this deep subject related to geometric invariant theory.
- I illustrate the general theory with a thorough discussion of a very simple example, cubics in $\mathbb{P}_1$, in which case the group is SL_2.
- As the computation of Molien series via the Weyl character formula involves contour integrals, I give a few tricks to compute residues of rational functions efficiently. Those tricks are crucial to get the job done without filling hundreds of gigabytes of memory. This is in fact the only place in these notes where I can mildly claim some originality.
- At last the case of cubics in $\mathbb{P}_3$, with action by SL_4, is tackled.

Section 2 is a tribute paid to the beauty of the geometry of non-singular cubic surfaces. Non-degenerate cubic curves have the amazing property that they are in fact Abelian groups in a natural way (again, see e.g. [14,17]). It turns out that non-degenerate cubic surfaces also have an amazing property: they contain 27 lines[4] whose incidence relations have a very rich combinatorial content. Though the discussions I had with Raymond did not touch this aspect, I decided to

[4] My first meeting with the 27 lines was during my PhD thesis, when I learned of their relationship with the $28 = 27 + 1$ bitangents to a non-singular plane quartic curve. This popped out while studying a question related to modular invariants for certain conformal field theories.

include some elementary computations that at least give a feeling of this beautiful theory. The level of the discussion will be that of a middlebrow little inspired[5] algebraic geometer of the end of the nineteenth century. Apart from [5,4], modern books that touch the subject of cubic surfaces, the 27 lines and their combinatorics are [11,13]. The modern approach is extremely powerful, but requires a serious investment. The elementary down-to-earth computations proposed in these notes have been a motivation for me to try to understand the more systematic but abstract methods of algebraic geometry. If the reader gets the same impetus, my goals will be fulfilled.

- I start with some very basic definitions, followed by an elementary counting argument explaining when hypersurfaces are expected to contain embedded linear subspaces: the criterion one gets involves the dimension and degree of the hypersurface, and the dimension of the linear subspace.
- This general counting suggests that there are lines on generic cubic surfaces, and I give a refined counting argument to suggest that generic cubic surfaces contain at least three lines, two of them being disjoint but intersected by the third.
- I then turn to (naive) deformation theory, in several steps. First, if a cubic surface contains a line and is non-singular along that line, then each infinitesimal deformation of this cubic surface contains a deformation of the line. Second, if a cubic surface contains two intersecting lines and is non-singular along these two lines, then the infinitesimal deformations of the two lines still intersect. However, when a cubic surface with three lines intersecting in a single point is subject to an infinitesimal deformation, the intersection point blows up in an infinitesimal triangle, unless the deformation satisfies one condition.
- The next step is a counting argument for the existence of 27 lines on a given non-singular cubic surface. This is followed by a detailed study of the Fermat cubic surface. I combine the explicit computations in that case with the previous deformation arguments to infer the general intersection pattern of the system of 27 lines, and I conclude with some remarks concerning the remarkable combinatorial features it carries via the so-called Schäfli double six.
- I conclude with the explanation why cubic surfaces are rational surfaces.

Appendix A is a reminder of basic algebraic homogeneous constructions (tensor products, symmetric and antisymmetric algebra), and their combinatorial features related to representation theory.

The beginning of Appendix B is a reminder on roots, weights and characters. It can be seen as a very explicit illustration for $SL_n(\mathbb{C})$ of the program suggested in Chapter 14 of [8]. The character formula then is used to give an explicit representation of the coefficients of the Molien series to be used in the main text. In particular, Proposition 23, found by Raymond in the literature [2] (and to be contrasted with Proposition 24, based on orthogonality of characters) proved crucial to simplify computations in the end.

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[5] This was more than enough at the time to confine him to lasting anonymity.

2. Cubic surfaces

Before embarking on the explicit study of cubic surfaces, we give some very basic definitions (projective space, hypersurface, non-singular hypersurface,...). For a general introduction to algebraic geometry, see e.g. [21,22].

2.1. Basic definitions

Recall that n dimensional projective space $\mathbb{P}_n$ is the space of $n+1$-tuples $(x_1,\cdots,x_{n+1})$, not all equal to 0, modulo global scalings. We shall always think of $x_1,\cdots,x_{n+1}$ as complex numbers, but many definitions go through when $\mathbb{C}$ is replaced by an arbitrary commutative field[6] $\mathbb{K}$. Formally:

Definition 1. $\mathbb{P}_n := \left(\mathbb{C}^{n+1}\setminus\{(0,\cdots,0)\}\right)/\mathcal{R}$ where $\mathcal{R}$ denotes the following equivalence relation. If $(x_1,\cdots,x_{n+1}),(y_1,\cdots,y_{n+1})\in\mathbb{C}^{n+1}\setminus\{(0,\cdots,0)\}$,

$$(x_1,\cdots,x_{n+1})\mathcal{R}(y_1,\cdots,y_{n+1})$$

holds if and only if there is scalar λ such that

$$(x_1,\cdots,x_{n+1})=\lambda(y_1,\cdots,y_{n+1}).$$

It is customary to denote by $(x_1;\cdots;x_{n+1})$ the equivalence class in $\mathbb{P}_n$ of $(x_1,\cdots,x_{n+1})\in\mathbb{C}^{n+1}\setminus\{(0,\cdots,0)$. If $x_{n+1}\neq 0$, $(x_1;\cdots;x_{n+1})$ has an unique representative of the form $(y_1,\cdots,y_n,1)$, namely $y_m=x_m/x_{n+1}$ for $m=1,\cdots,n$. This is a copy of $\mathbb{C}^n$ inside $\mathbb{P}_n$, and what remains is naturally identified with $\mathbb{P}_{n-1}$. This can be used in two ways: (i) Splitting $\mathbb{P}_n$ according to the number of terminal zeros, we decompose it as the disjoint union of $\mathbb{C}^n,\mathbb{C}^{n-1},\cdots,\mathbb{C}^0$. (ii) The last coordinate plays no special role, and one can also view $\mathbb{P}_n$ as obtained by patching together in an appropriate way $n+1$ copies of $\mathbb{C}^n$. The change of variable to go from one patch to the other is smooth, and $\mathbb{P}_n$ is a nice object of dimension n.

Definition 2. A hypersurface Σ of degree d in $\mathbb{P}_n$ is the zero locus of a non-zero homogeneous polynomial $P_{n+1,d}$ of degree d in $n+1$ variables, i.e. the set of points $(x_1;\cdots;x_{n+1})\in\mathbb{P}_n$ such that $P_{n+1,d}(x_1,\cdots,x_{n+1})=0$.

Note that as $P_{n+1,d}$ is homogeneous, the global scaling of the variables is irrelevant in this relation and the definition is meaningful. By naïve counting, Σ has dimension $n-1$ i.e. codimension 1, hence the name hypersurface.[7]

Definition 3. Σ is non-singular at $(x_1;\cdots;x_{n+1})$ if $P_{n+1,d}(x_1,\cdots,x_{n+1})=0$ but $\frac{\partial P_{n+1,d}}{\partial X_m}\neq 0$ at $(x_1,\cdots,x_{n+1})$ for at least one $m\in\{1,\cdots,n+1\}$. One says that Σ is non-singular if it is non-singular everywhere.

The reader may check that this reflects the standard notion of smoothness.[8]

[6] It helps sometimes to assume that $\mathbb{K}$ is algebraically closed and of characteristic 0.

[7] To do the counting once: $n-1=(n+1)-1-1$, i.e. $n+1$ variables, -1 for the relation and -1 for the scaling.

[8] The only question is homogeneity.

Our main interest in what follows is the case $(n+1, d) = (4, 3)$: a cubic surface S is the zero locus of a homogeneous polynomial $P_{4,3}$ of degree 3 in 4 variables which we choose to be W, X, Y, Z in what follows.

Our aim is to motivate one of the cornerstones of classical algebraic geometry: a non-singular cubic surface contains 27 lines.

2.2. Hypersurfaces containing linear subspaces

Why are there lines on generic cubic surfaces[9]? We do a more general computation which is heuristic but gives a useful guideline. The main computation is closely related to the approach in [12].

Definition 4. If $(a_1, \cdots, a_{n+1})$ is a non-zero $n+1$-tuple, the zero locus H of the linear equation $a_1 X_1 + \cdots a_{n+1} X_{n+1} = 0$ in $\mathbb{P}_n$ is called an hyperplane in $\mathbb{P}_n$.

There are plenty of linear bijections between H and $\mathbb{P}_{n-1}$. More generally, a k-plane in $\mathbb{P}_n$ is the intersection of $n-k$ independent hyperplanes (i.e. the $n-k$ vectors $(a_1, \cdots, a_{n+1})$ defining the hyperplanes are linearly independent). In particular a line in $\mathbb{P}_n$ is the intersection of $n-1$ (independent) hyperplanes. We start with some elementary parameter counting.

Proposition 1. *The family of k-planes in $\mathbb{P}_n$ has dimension*[10] $\delta_{n+1,k} := (n-k)(k+1)$ *i.e. depends on* $(n-k)(k+1)$ *parameters.*

We shall in fact "prove" more. Before starting, let us note that the space of k-planes in $\mathbb{P}_n$ is an important algebraic object called a Grassmanian and denoted by $G(n, k)$. We shall explain why $G(n, k)$ is a nice object of dimension $(n-k)(k+1)$ and how equations for it can be written down via the so-called Plücker embedding.

We describe a given k-plane as the intersection of $n-k$ hyperplanes. Thus we look for the space of solutions (modulo scaling) of a linear system

$$\sum_{m=1}^{n+1} a_l^m X_m = 0 \text{ for } l = 1, \cdots, n-k$$

of maximal rank. We let $H(a)$ denote this space.

Elementary linear algebra tells us that the rank is maximal if and only if at least one of the square sub-matrices of $(a_l^m)_{l=1,\cdots n-k\ m=1,\cdots n+1}$ obtained by retaining exactly $n-k$ columns has a non-vanishing determinant. If J is a subset of $\{1, \cdots, n+1\}$ of size $n-k$ we let $V(a)_J$ denote the determinant involving the columns listed in J.

Elementary linear algebra tells us also that if the solution set (modulo scaling) of $\sum_1^{n+1} b^m X_m = 0$ contains $H(a)$ then one can write $b^m = \sum_{l=1}^{n-k} c^l a_l^m$ for $m = 1, \cdots, n+1$ for uniquely defined coefficients c^l. This implies that if $\sum_{m=1}^{n+1} a_l^m X_m = 0$ for $l = 1, \cdots, n-k$ and $\sum_{m=1}^{n+1} b_l^m X_m = 0$ for $l = 1, \cdots, n-k$ describe the same k-plane, i.e. if $H(b) = H(a)$, there is an invertible matrix $c_{l'}^l$ such that $b_{l'}^m = \sum_{l=1}^{n-k} c_{l'}^l a_l^m$ for $l = 1, \cdots, n-k$ and $m = 1, \cdots, n+1$.

[9] Though the meaning of this question is intuitively clear, see below in case of doubt.

[10] Warning: this is not a Kronecker delta!

Then $V(b)_J = V(a)_J \det c$ for every $n-k$-subset J of $\{1,\cdots,n+1\}$. Thus, one can assign to each k-plane H a well-defined point in the projective space $\mathbb{P}_{\binom{n+1}{n-k}-1}$: the collections of the $V(a)_J$ modulo scaling for any choice of $(n-k)\times(n+1)$ matrix a such that $H = H(a)$. It is easy to see that this map is one-to-one. Take J such that $V(a)_J \neq 0$, and to simplify notation suppose that $J = \{1,\cdots,n-k\}$. Take $c^l_{l'}$ to be the inverse of $a^l_{l'}$ $(l,l' \in \{1,\cdots,n-k\})$. The equations for $H(a)$ then take the form $X_l + \sum_{m=n-k+1}^{n+1} b_l^m X_m = 0$ for $l = 1,\cdots,n-k$.

This form shows two things: (i) The set of k-planes has dimension $(n-k)(k+1)$, and we have produced explicit coordinates valid when $V(a)_J \neq 0$, and (ii)

$$b_l^m = (-)^{l+k-n} V(b)_{(J\setminus\{l\})\cup m} = (-)^{l+k-n} \frac{V(a)_{(J\setminus\{l\})\cup m}}{V(a)_J}$$

so that equations for $H(a)$ can be retrieved from the $V(a)$s.

Of course $J = \{1,\cdots,n-k\}$ plays no special role, any other J such that $V(a)_J \neq 0$ would do. In case several $V(a)_J$s are non-zero, one can go smoothly from one set of coordinates to the other, and we conclude from the first point that the space of k-planes in $\mathbb{P}_n$ is a smooth object obtained by patching together in an appropriate way $\binom{n-k}{n+1}$ copies of $\mathbb{C}^{(n-k)(k+1)}$.

The second point leads to a more global viewpoint. For $k=0$ one recovers that the set of points in $\mathbb{P}_n$ is $\mathbb{P}_n$ itself, no big surprise. For $k = n-1$ one recovers that the set of hyperplanes in $\mathbb{P}_n$ is a copy of $\mathbb{P}_n$, a manifestation that a finite dimensional vector space and the dual space of linear forms on it have the same dimension. As soon as $k \neq 1, n-1$, the $V(a)$s satisfy relations. To illustrate this fundamental fact discovered by Plücker, we look at the case $(n,k) = (3,1)$, lines in $\mathbb{P}_3$, which appears naturally in the discussion of lines on cubic surfaces. We start from

$$a := \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{pmatrix}$$

which we normalize, assuming that $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \neq 0$ to get

$$b := \begin{pmatrix} 1 & 0 & b_{1,3} & b_{1,4} \\ 0 & 1 & b_{2,3} & b_{2,4} \end{pmatrix}.$$

Now $V(b)_{1,2} = 1$, $V(b)_{1,3} = b_{2,3}$, $V(b)_{1,4} = b_{2,4}$, $V(b)_{2,3} = -b_{1,3}$, $V(b)_{2,4} = -b_{1,4}$, in agreement with the general formulae given above, and $V(b)_{3,4} = b_{1,3}b_{2,4} - b_{2,3}b_{1,4}$. Thus

$$V(b)_{1,2}V(b)_{3,4} = -V(b)_{1,4}V(b)_{2,3} + V(b)_{1,3}V(b)_{2,4}.$$

This relation is homogeneous, so it holds for a if it holds for b (a direct verification is easy too) and we have a compact description of the space of lines in $\mathbb{P}_3$ as the quadric

$$V_{1,2}V_{3,4} - V_{1,3}V_{2,4} + V_{1,4}V_{2,3} = 0$$

in $\mathbb{P}_5$.

Proposition 2. *The space of homogeneous polynomials of degree d in $n+1$ variables has dimension $\Delta_{n+1,d} := \frac{(n+d)!}{n!d!}$.*

This is familiar to physicists as a counting of bosonic states and can be retrieved by an easy recursive argument as follows. To simplify the notation, in this discussion we denote by

$P_{m,d} := P_{m,d}(X_1, \cdots, X_m)$ a polynomial of degree d in $X_1, \cdots, X_m$, i.e. we omit the explicit mention of the variables. A homogeneous polynomial of degree d in $n+1$ variables $P_{n+1,d}$ can be decomposed uniquely as

$$P_{n+1,d} = X_{n+1} P_{n+1,d-1} + P_{n,d}.$$

Explicitly, $P_{n,d}(X_1, \cdots, X_n) = P_{n+1,d}(X_1, \cdots, X_n, 0)$. Thus

$$\Delta_{n+1,d} = \Delta_{n+1,d-1} + \Delta_{n,d} \text{ for } n \geq 1 \text{ and } d \geq 1.$$

Together with the boundary conditions $\Delta_{n,0} = 1$ and $\Delta_{1,d} = 1$, this implies that $\Delta_{n+1,d}$ is equal to the binomial coefficient $\frac{(n+d)!}{n!d!}$ as announced.

This leads to the following:

Proposition 3. *(Only a rule of thumb in fact) The condition for a generic hypersurface of degree d in $\mathbb{P}_n$ to contain a k-plane is*

$$\delta_{n+1,k} - \Delta_{k+1,d} = 0.$$

Let $P_{n+1,d}$ be a non-zero homogeneous polynomial of degree d in $n+1$ variables. The zero locus $P_{n+1,d}(x_1, \cdots, x_{n+1}) = 0$ defines an hypersurface Σ in $\mathbb{P}_n$.

By a linear transformation any k-plane in $\mathbb{P}_n$ can be brought to the k-plane $\Lambda : \ X_{n+1} = X_n = \cdots = X_{k+2} = 0$.

But $P_{n+1,d}$ can be written uniquely as

$$P_{n+1,d} = X_{n+1} P_{n+1,d-1} + X_n P_{n,d-1} + \cdots + X_{k+2} P_{k+2,d-1} + P_{k+1,d}.$$

Then Λ is on Σ if and only if $P_{k+1,d}(X_1, \cdots, X_{k+1})$ vanishes identically. Hence, the dimension of the space of polynomials $P_{n+1,d}(X_1, \cdots, X_{n+1})$ vanishing on Λ is $\Delta_{n+1,d-1} + \Delta_{n,d-1} + \cdots + \Delta_{k+2,d-1} = \Delta_{n+1,d} - \Delta_{k+1,d}$. Thus we have to compare $\Delta_{n+1,d}$, the total dimension, to $\delta_{n+1,k} + \Delta_{n+1,d} - \Delta_{k+1,d}$, the dimension of the subspace vanishing on a k-plane. This concludes the argument.

To summarize, the existence of k-planes included in a generic hypersurface of degree d in $\mathbb{P}_n$ depends on the value of $\delta_{n+1,k} - \Delta_{k+1,d} = (n-k)(k+1) - \frac{(k+d)!}{k!d!}$. If $\delta_{n+1,k} - \Delta_{k+1,d}$ is < 0, one expects that only special hypersurfaces of degree d in $\mathbb{P}_n$ contain k-planes, whereas if $\delta_{n+1,k} = \Delta_{k,d}$ one expects that a generic hypersurface of degree d in $\mathbb{P}_n$ contains k-planes.

This counting, which is basically correct, must nevertheless be taken with a grain of salt because in fact $\delta_{n+1,k} - \Delta_{k+1,d}$ can also be > 0, which means that some over-counting is possible. But the over-counting might also occur when $(n-k)(k+1) - \frac{(k+d)!}{k!d!} = 0$. One can show that for $d > 3$ our naïve results are correct, but it is known that for $d = 2$ (where the discussion boils down to the sign of $2(n-1) - 3k$) exceptions do occur: for instance one can show that there are no planes ($k = 2$) on generic quadrics in five variables ($n = 4$) though $2 \times (4-1) - 3 \times 2 = 0$. See [12] for a discussion.

Anyway, the equality for $n = 3$, $d = 3$, $k = 1$ (both sides are equal to 4) is the result we were after, that a generic cubic surface contains a line.

This relation holds for instance also for $n = 4$, $d = 5$, $k = 1$ (lines on quintics in 5 variables, both sides are equal to 6; this is in fact another exceptional case, because there is an infinite number of lines), $n = 7$, $d = 4$, $k = 2$ (planes on quartics in 8 variables, both sides are equal to 15), and so on.

2.3. A refined counting

The counting argument in the previous section supported the fact that a generic cubic surface contains a line. In fact, we shall see that counting even suggests:

Proposition 4. *A generic cubic surface contains (at least) three lines L, M, N such that N intersects L and M but L and M are disjoint.*

Our "proof" is again based on using linear transformations to get a workable form for the equation of a generic surface.

Let us look at the space of cubic surfaces containing two non-intersecting lines L and M and a third line N intersecting both L and M. The lines L and N define a plane, which we can take to be $W = 0$. The lines M and N define a plane, which we can take to be $Z = 0$. Thus $N : W = Z = 0$. The second equation for L can be taken to be $X = 0$ and the second equation for M can be taken to be $Y = 0$. The 10 monomials

$$W^2Z, W^2Y, WXY, WXZ, WY^2, WYZ, WZ^2, X^2Z, XYZ, XZ^2$$

all vanish on L, M, N, while no non-trivial combination of the other 10 monomials

$$W^3, W^2X, WX^2, X^3, X^2Y, XY^2, Y^3, Y^2Z, YZ^2, Z^3$$

can vanish on L, M, N (verification is immediate). Thus, the homogeneous polynomials of degree 3 in 4 variables vanishing on L, M, N form a 10-dimensional vector space. The general continuous transformations preserving the lines L, M, N form a 6-dimensional family, represented by a pair of triangular 2×2 matrices: $(W, X) \to (aW, bW + cX)$ and $(Y, Z) \to (dY + eZ, fZ)$ (with ac and df non-zero), while the non-singular 4×4 matrices form a 16-dimensional family, so that the action of general non-singular linear transformations on the 10-dimensional family of homogeneous polynomials of degree 3 in 4 variables vanishing on L, M, N is a $10 + 16 - 6 = 20$-dimensional family. But 20 is the dimension of the space of all polynomials of degree 3 in 4 variables. Thus a generic cubic surface should carry a configuration of three lines with the same incidence properties as L, M, N.

Again, such a counting must be taken with a grain of salt, but it is encouraging, and the result is confirmed by a more rigorous analysis.

2.4. Lines and intersections of lines on cubic surfaces

Another way to show the plausibility of the existence of lines on cubic surfaces is by deformation arguments: if a cubic surface contains a line, its infinitesimal deformations also do. Then it becomes more or less possible to follow the deformations of lines when the surface changes, and to prove the generic existence of lines just by finding a line on "one" cubic surface. More precisely, we shall assume that the existence of deformations to first order in perturbation theory, if non-degenerate, guarantees the existence of finite deformations. We shall make the meaning of this statement more precise in what follows. Let us simply stress that such a naïve approach is a valuable starting point even for modern algebraic geometry. . . but only a starting point.

Proposition 5. *If a cubic surface S containing a line L, and non-singular along L, is submitted to an infinitesimal deformation, the deformed surface still contains a line which is an infinitesimal deformation of L.*

We assume that some cubic surface S, described as the zero locus of a homogeneous polynomial of degree 3 in 4 variables W, X, Y, Z, contains a line L. As already observed in the discussion of the previous section, by a linear transformation, we may assume that this line is the intersection of the planes $Y = 0$ and $Z = 0$. Thus the cubic surface is the zero locus of a homogeneous polynomial of the restricted form

$$P_{4,3} = Z P_{4,2} + Y P_{3,2}(W, X, Y).$$

If $Q_{4,3}$ is an arbitrary homogeneous polynomial of degree 3 in W, X, Y, Z, we look for a line on $P_{4,3} + \varepsilon Q_{4,3} = 0$. A generic first order deformation of the line $Y = Z = 0$ may be written as $Y - \varepsilon(aW + bX) = Z - \varepsilon(cW + dX) = 0$. Expressing that the deformed line is on the deformed surface to first order in ε yields

$$(cW + dX)P_{4,2}(W, X, 0, 0) + (aW + bX)P_{3,2}(W, X, 0) = Q_{4,3}(W, X, 0, 0).$$

We can reinterpret this equation as follows. The left-hand side is a linear map from the space of deformations of the line $Y = Z = 0$ (4 parameters a, b, c, d) to the space of homogeneous polynomials of degree 3 in W, X (4 parameters) and the question is whether this map is onto. Because dimensions are the same, the map is onto if and only if it is one-to-one. As $P_{4,2}(W, X, 0, 0)$ and $P_{3,2}(W, X, 0)$ are of degree two, the existence of a nontrivial 4-tuple a, b, c, d such that $(cW + dX)P_{4,2}(W, X, 0, 0) + (aW + bX)P_{3,2}(W, X, 0) = 0$ is equivalent to the fact that $P_{4,2}(W, X, 0, 0)$ and $P_{3,2}(W, X, 0)$ have a common factor. Now observe that

$$\frac{\partial P_{4,3}}{\partial W}(W, X, 0, 0) = 0 \quad \frac{\partial P_{4,3}}{\partial X}(W, X, 0, 0) = 0$$

while

$$\frac{\partial P_{4,3}}{\partial Y}(W, X, 0, 0) = P_{3,2}(W, X, 0) \quad \frac{\partial P_{4,3}}{\partial Z}(W, X, 0, 0) = P_{4,2}(W, X, 0, 0).$$

The existence of a point along the line $Y = Z = 0$ where all the partial derivatives vanish is thus equivalent to the fact that $P_{4,2}(W, X, 0, 0)$ and $P_{3,2}(W, X, 0)$ have a common 0 i.e. a common factor.

Thus we have proved: suppose that a cubic surface S contains a line L; then an arbitrary infinitesimal deformation of S contains a single line which is an infinitesimal deformation of L if and only if S is non-singular along L.

As the example of the cubic Fermat surface $W^3 + X^3 + Y^3 + Z^3 = 0$ detailed below will illustrate, lines on cubics may intersect. So we go one step further and assume that S contains 2 lines L and L' intersecting at a point P.

Proposition 6. *If a cubic surface S containing two intersecting lines L and L', and non-singular along L and L', is submitted to an infinitesimal deformation, the deformed surface still contains two intersecting lines which are infinitesimal deformations of L and L'.*

Once again, by an appropriate linear transformation, we may assume that L has equations $Y = Z = 0$ and L' has equations $X = Z = 0$. Then L and L' intersect at $P = (1; 0; 0; 0)$ and a defining polynomial for S decomposes as

$$P_{4,3}(W, X, Y, Z) = Z P_{4,2}(W, X, Y, Z) + XY P_{3,1}(W, X, Y).$$

We assume that S is non-singular along L and L'. The question is whether after infinitesimal deformation of S the deformations of L and L' still do intersect. We keep the previous notation for the deformation of L and write the deformation of L' as $X - \varepsilon(a'W + b'Y) = Z - \varepsilon(c'W + d'Y) = 0$ so that the equations are

$$(cW + dX)P_{4,2}(W, X, 0, 0) + (aW + bX)XP_{3,1}(W, X, 0) = Q_{4,3}(W, X, 0, 0),$$
$$(c'W + d'Y)P_{4,2}(W, 0, Y, 0) + (a'W + b'Y)YP_{3,1}(W, 0, Y) = Q_{4,3}(W, 0, Y, 0).$$

By the above discussion, as S is assumed to be non-singular along L (resp. L'), $P_{4,2}(W, X, 0, 0)$ (resp. $P_{4,2}(W, 0, Y, 0)$) cannot contain X (resp. Y) as a factor. Thus, letting $X = 0$ in the first equation and $Y = 0$ in the second, we get

$$c = c' = \frac{Q_{4,3}(1, 0, 0, 0)}{P_{4,2}(1, 0, 0, 0)}.$$

This is enough to guarantee that the deformations of L and L' intersect, to first order in ε, at $(1; \varepsilon a'; \varepsilon a; \varepsilon c)$.

We also remark:

Proposition 7. *If a cubic surface S contains two intersecting lines L and L', it contains a third L'' intersecting both L and L', and such that L, L', L'' are coplanar.*

With the above choices for the equations for L and L', the equations for L'' can be taken to be $Z = P_{3,1}(W, X, Y) = 0$. Note that $P_{3,1} \equiv 0$ would lead to split S as the union of the plane $Z = 0$ and the quadric $P_{4,2}(W, X, Y, Z) = 0$, i.e. to singular points. Two possibilities remain: (i) $P_{3,1}(W, X, Y)$ involves the W variable, and then after a linear transformation we can assume that $P_{3,1}(W, X, Y)$ is proportional to W, so the equation for S takes the form $ZP_{4,2}(W, X, Y, Z) + pWXY = 0$ for some non-zero p; the lines L, L', L'' intersect in 3 distinct points, i.e. they define a non-degenerate triangle; or (ii) $P_{3,1}(W, X, Y) = qX + rY$ for some non-zero q and r; then L, L', L'' meet at the point $(1; 0; 0; 0)$. Such a point at which three lines on a cubic surface meet is called an Eckhard point

Proposition 8. *Triple intersection point, i.e. Eckhard points, do not survive generic deformatisons.*

Indeed, write the equations for the deformation of L'' as $qX + rY - \varepsilon(a''W + \cdots) = Z - \varepsilon(c''W + \cdots) = 0$ where the $\cdots$ denote any linear combination of X and Y linearly independent of $qX + rY$. We already know that $c = c' = c''$ and the three deformed intersection points are $L \cap L' : (1; \varepsilon a'; \varepsilon a; \varepsilon c)$, $L \cap L'' : (1; \varepsilon a'; \varepsilon(a'' - qa')/r; \varepsilon c)$ and $L' \cap L'' : (1; \varepsilon(a'' - ra)/q; \varepsilon a; \varepsilon c)$. These three points coïncide if and only if $a'' = qa' + ra$. On the other hand, some algebra shows that a, a', a'' can be computed from the sole knowledge of $P_{4,2}(0, X, Y, 0)$ and $\frac{\partial Q_{4,3}}{\partial W}(0, X, Y, 0)$. Explicitly

$$cP_{4,2}(0, X, 0, 0) + aqX^2 = \frac{\partial Q_{4,3}}{\partial W}(0, X, 0, 0),$$
$$cP_{4,2}(0, 0, Y, 0) + a'rY^2 = \frac{\partial Q_{4,3}}{\partial W}(0, 0, Y, 0),$$
$$cP_{4,2}(0, X, -\frac{q}{r}X, 0) - a''\frac{q}{r}X^2 = \frac{\partial Q_{4,3}}{\partial W}(0, X, -\frac{q}{r}X, 0).$$

Rewriting the second equality as

$$cP_{4,2}(0,0,-\frac{q}{r}X,0)+a'\frac{q^2}{r}X^2=\frac{\partial Q_{4,3}}{\partial W}(0,0,-\frac{q}{r}X,0),$$

we infer that the relation $a''=qa'+ra$ holds if and only if

$$c\left(P_{4,2}(0,1,0,0)+P_{4,2}(0,0,-\frac{q}{r},0)+P_{4,2}(0,1,-\frac{q}{r},0)\right)$$
$$=\frac{\partial Q_{4,3}}{\partial W}(0,1,0,0)+\frac{\partial Q_{4,3}}{\partial W}(0,0,-\frac{q}{r},0)+\frac{\partial Q_{4,3}}{\partial W}(0,1,-\frac{q}{r},0).$$

Remember that $c=\frac{Q_{4,3}(1,0,0,0)}{P_{4,2}(1,0,0,0)}$, so the above is a non-empty constraint. Thus a single constraint has to be satisfied in order for a given Eckhard point to survive an infinitesimal deformation.

Taking again for granted that infinitesimal deformations can be integrated to yield finite ones, we infer that the class of smooth cubic surfaces with Eckhard points is singled out by one relation among the coefficients of the cubic.

2.5. Summary on lines and their intersections

Our first order computation plus the usual facts about perturbation theory in finite dimension guarantee that if some cubic surface S contains a line L along which S is non-singular, then a finite but sufficiently small[11] deformation S' of S will still carry a line L' which is a deformation of L along which S' is non singular. In particular, if S is non-singular everywhere, a finite but sufficiently small deformation S' of S will remain smooth and the lines of S and S' are in natural one-to-one correspondence. Also, their incidence relations remain the same. But possible threefold intersections are accidental.

Then we could go on with a version of analytic continuation and ask what happens under larger deformations. If we can go from one non-singular cubic surface S to another one S' by a succession of small deformations, the lines on S can be followed along the way to those of S' and the incidence relations are preserved. But it might be that two different ways to go from S to S' lead to a different correspondence between lines.

The crucial question is then whether any 2 non-singular cubic surfaces can be connected by a sequence of small deformations (within the space of non-singular cubic surfaces). The answer is yes.[12] Thus all non-singular cubic surfaces carry the same number of lines and their incidence relations are the same.

It remains to count the number of lines. We start be a general perturbation argument which reinforces the intuition that the number of lines is the same for all non-singular cubic surfaces. But if this is taken for granted, we could also simply understand the lines on a single suitable cubic surface and we do that next for the Fermat cubic surface $W^3+X^3+Y^3+Z^3=0$.

2.6. Why 27?

A simple counting argument goes as follows. The surface $XYZ=0$ is singular (it is the union of 3 planes, it contains infinitely many lines and is singular along the 3 intersection lines). However a small generic perturbation will make it non-singular. So we look at the limit of lines

[11] This is where some questions arise: should one use the complex topology, or possibly a topology better suited to algebraic geometry? The author has all reasons to remain silent on these subtleties.

[12] But we shall not try to give even a vague argument. See [5].

on $XYZ + \varepsilon P_{3,3}(W, X, Y, Z) = 0$. In the limit, a generic line will be close to one of the planes $X = 0$, $Y = 0$ or $Z = 0$. Let us look at the last possibility. Write the line, to first order in ε, as $W - (aX + bY) = Z - \varepsilon(cX + dY) = 0$. This leads to

$$XY(cX + dY) + P_{3,3}(aX + bY, X, Y, 0) = 0$$

Let $Y = 0$ (resp. $X = 0$) to get $P_{3,3}(a, 1, 0, 0) = 0$ (resp. $P_{3,3}(b, 0, 1, 0) = 0$). These are two equations for a and b, generically of degree 3, leading to 9 possibilities. For each of these, c and d are determined as the coefficients of X^2Y and XY^2 in $-P_{3,3}(aX + bY, X, Y, 0)$. Thus generically 9 lines on $XYZ + \varepsilon P_{3,3}(W, X, Y, Z) = 0$ have a limiting position in the plane $Z = 0$. By the same argument 9 lines have a limiting position in the plane $X = 0$ and in the plane $X = 0$ for a total of 27 lines on a generic cubic surface.

Though we are far from a real proof, we have collected enough evidence to motivate the following theorem:

Theorem 1. *Every non-singular cubic surface contains exactly* 27 *lines.*

Our next task is to study in detail a special case.

2.7. The Fermat cubic surface

The 27 lines are easy to spot on certain special cubic surfaces.

One example is the cubic Fermat surface[13]:

$$F_3 : \ W^3 + X^3 + Y^3 + Z^3 = 0.$$

The cubic Fermat surface has many symmetries: arbitrary permutation of the coordinates together with multiplication of each coordinate by an independent cubic root of unity. In the sequel, we let ξ be a primitive cubic root of unity, so that $\xi^2 + \xi + 1 = 0$, i.e. $\xi := e^{\pm 2i\pi/3}$.

From the identity $a^3 + b^3 = (a + b)(a + \xi b)(a + \xi^2 b)$ one immediately spots a number of lines on F_3. At the risk of being boring, we give a complete list:

$$\begin{array}{rl}
W + X = 0 & Y + Z = 0 \\
W + X = 0 & Y + \xi Z = 0 \\
W + X = 0 & Y + \xi^2 Z = 0 \\
W + \xi X = 0 & Y + Z = 0 \\
W + \xi X = 0 & Y + \xi Z = 0 \\
W + \xi X = 0 & Y + \xi^2 Z = 0 \\
W + \xi^2 X = 0 & Y + Z = 0 \\
W + \xi^2 X = 0 & Y + \xi Z = 0 \\
W + \xi^2 X = 0 & Y + \xi^2 Z = 0
\end{array}$$

[13] The name is inherited from the situation with 3 variables, i.e. the celebrated Fermat curves.

$$\begin{array}{rl}
W+Y=0 & X+Z=0 \\
W+Y=0 & X+\xi Z=0 \\
W+Y=0 & X+\xi^2 Z=0 \\
W+\xi Y=0 & X+Z=0 \\
W+\xi Y=0 & X+\xi Z=0 \\
W+\xi Y=0 & X+\xi^2 Z=0 \\
W+\xi^2 Y=0 & X+Z=0 \\
W+\xi^2 Y=0 & X+\xi Z=0 \\
W+\xi^2 Y=0 & X+\xi^2 Z=0 \\
W+Z=0 & X+Y=0 \\
W+Z=0 & X+\xi Y=0 \\
W+Z=0 & X+\xi^2 Y=0 \\
W+\xi Z=0 & X+Y=0 \\
W+\xi Z=0 & X+\xi Y=0 \\
W+\xi Z=0 & X+\xi^2 Y=0 \\
W+\xi^2 Z=0 & X+Y=0 \\
W+\xi^2 Z=0 & X+\xi Y=0 \\
W+\xi^2 Z=0 & X+\xi^2 Y=0
\end{array}$$

Note that the automorphism group of F_3 acts transitively on the lines: in plain language, any line in the list can be obtained from the first one $W+X=0$, $Y+Z=0$ by some appropriate permutation of the variables and multiplication by cubic roots of unity. However, even if individual lines are equivalent, pairs of lines are not. Some do intersect and some don't.

To get a better description of the pattern of intersections among lines, let us list the ones intersecting the line $W+X=0$, $Y+Z=0$. We find 10 that split in 5 subsets of 2 as follows:

$$W+X=0,\, Y+\xi Z=0 \text{ and } W+X=0,\, Y+\xi^2 Z=0$$

intersecting at $(1;-1;0;0)$ which also belongs to $W+X=0$, $Y+Z=0$, the three lines are contained in the plane $W+X=0$;

$$W+\xi X=0,\, Y+Z=0 \text{ and } W+\xi^2 X=0,\, Y+Z=0$$

intersecting at $(0;0;1;-1)$ which also belongs to $W+X=0$, $Y+Z=0$, the three lines are contained in the plane $Y+Z=0$;

$$W+Y=0,\, X+Z=0 \text{ and } W+Z=0,\, X+Y=0$$

intersecting at $(1;1;-1;-1)$ but intersecting the original line at $(1;-1;-1;1)$ and $(1;-1;1;-1)$ respectively, leading to a triangle contained in the plane $W+X+Y+Z=0$;

$$W + \xi Y = 0, X + \xi Z = 0 \text{ and } W + \xi Z = 0, X + \xi Y = 0$$

intersecting at $(1; 1; -\xi^2; -\xi^2)$ but intersecting the original line at $(1; -1; -\xi^2; \xi^2)$ and $(1; -1; \xi^2; -\xi^2)$ respectively, leading to a triangle contained in the plane $W + X + \xi Y + \xi Z = 0$;

$$W + \xi^2 Y = 0, X + \xi^2 Z = 0 \text{ and } W + \xi^2 Z = 0, X + \xi^2 Y = 0$$

intersecting at $(1; 1; -\xi; -\xi)$ but intersecting the original line at $(1; -1; -\xi; \xi)$ and $(1; -1; \xi; -\xi)$ respectively, leading to a triangle contained in the plane $W + X + \xi^2 Y + \xi^2 Z = 0$.

Lines in different subsets do not intersect. So we find that each line carries 2 points of triple intersection. Thus we find a total of 18 Eckhard points, where 3 lines meet. Also, each line participates to 3 triangles, for a total of 27 triangles.

Both Eckhard points and triangles correspond to triples of lines each of which intersects the two others. The Eckhard points correspond to degenerate triangles.

Now, we concentrate on the 16 lines that do not intersect $W + X = 0, Y + Z = 0$. They fall in 2 families: 4 are obtained from the $W + \xi X = 0, Y + \xi Z = 0$ by automorphisms of F_3 fixing $W + X = 0, Y + Z = 0$, while 12 are obtained in the same way from $W + \xi Y = 0$, $X + \xi^2 Z = 0$. We find that there are 5 lines intersecting both $W + X = 0$, $Y + Z = 0$ and $W + \xi X = 0$, $Y + \xi Z = 0$, namely

$$W + \xi X = 0, Y + Z = 0 \quad W + X = 0, Y + \xi Z = 0$$
$$W + Y = 0, X + Z = 0 \quad W + \xi Y = 0, X + \xi Z = 0 \quad W + \xi^2 Y = 0, X + \xi^2 Z = 0.$$

We also find that there are 5 lines intersecting both $W + X = 0$, $Y + Z = 0$ and $W + \xi Y = 0$, $X + \xi^2 Z = 0$, namely

$$W + X = 0, Y + \xi Z = 0 \quad W + \xi^2 X = 0, Y + Z = 0 \quad W + Z = 0, X + Y = 0$$
$$W + \xi Y = 0, X + \xi Z = 0 \quad W + \xi^2 Y = 0, X + \xi^2 Z = 0.$$

Note the following. Any 2 distinct intersecting lines define a unique plane and a unique (possibly degenerate) triangle. Thus, if L and M are non-intersecting lines and if L, L', L'' defines a (possibly degenerate) triangle, at most one among L' and L'' may intersect M. Thus, 5 is a priori the maximum number of lines intersecting both L and M. The above computation shows that this maximum is achieved.

We can now build on our detailed understanding of the Fermat cubic surface to draw general conclusions for generic cubic surfaces.

As we have seen above, Eckhard points, i.e. threefold intersections, do not survive perturbation, but twofold intersections do. If the Fermat cubic surface is perturbed a little bit, the 18 Eckhard points generically turn into small triangles, so that the total number of triangles on a generic cubic surface is $27 + 18 = 45$.

To summarize, each line on a generic cubic surface meets 10 other lines. Each of these 10 lines meets exactly one other amongst the 10, leading to 5 triangles one of whose sides is on the original line, for a total of 45 (possibly degenerate) triangles.

There are also non-intersecting lines on a non-singular surface. Exactly 5 lines on the surface joint two 2 given non-intersecting lines, and these 5 lines do not intersect each other.

The next subsection elaborates on these observations and deals with some combinatorial aspects of the configuration of the 27 lines, without reference to the underlying surface.

2.8. The Schäfli double six

Now let L_1, M_1 be 2 disjoint lines on a generic cubic surface. We know that 10 lines intersect L_1 and among them 5 do and 5 do not intersect M_1. More precisely, the 10 lines can be split in 5 subsets of 2 lines. The lines in each subset intersect each other, but do not intersect the 8 others, and exactly 1 in each subset intersects M_1. Of course, the roles of L_1 and M_1 can be interchanged.

Let M be a line intersecting L_1 but not M_1. The lines M and L_1 define a plane containing a third line, say N, which must meet both L_1 and M_1. Then N and M_1 define a plane containing a third line, which intersects M_1 but cannot intersect L_1, and we denote it by L. Applying the same argument in the opposite direction, we go back from L to M and thus there is a natural one-to-one correspondence between lines intersecting L_1 but not M_1 and lines intersecting M_1 but not L_1. Observe that L and M must be disjoint: they both intersect N so if they would meet, the three lines L, M, N would be coplanar. But L_1, M, N and L, M_1, N are also coplanar, and so the lines L_1 and M_1 would be coplanar as well, contradicting the fact that they do not intersect. Observe also that N is the only line intersects each of the 4 lines L, L_1, M, M_1.

We now concentrate on the lines intersecting both L and L_1. There are 5 of them, to be chosen among the 10 that intersect L_1. Among these 10, 5 intersect both L_1 and M_1, but among these 5, only N intersects L. Thus among the 5 lines that intersect L_1 but not M_1, 4 must intersect L. We know that L and M are disjoint, and this means that L must intersect the remaining 4 lines that intersects L_1 but not M_1.

Thus if $M_2, \cdots, M_6$ denote the 5 lines intersecting L_1 but not M_1, and $L_2, \cdots, L_6$ the 5 lines intersecting M_1 but not L_1, the labeling reflecting the natural one-to-one correspondence between these two sets of lines, we arrive at the following incidence relations: (i) The lines $L_1, \cdots, L_6$ are disjoint, (ii) The lines $M_1, \cdots, M_6$ are disjoint, (iii) The intersection $L_j \cap M_k$ is empty if $j = k$ and is a point if $j \neq k$.

Then $L_1, \cdots, L_6$ and $M_1, \cdots, M_6$ form a so-called Schäfli double six. Though L_1 and M_1 seemed to play a special role in the beginning, the incidence relations are completely symmetric under permutation of the labels.

For each j there are 5 lines intersecting both L_j and M_j, but each line is counted twice when j runs from 1 to 6. Indeed, the construction above shows that for each $j \neq k$ there is a single line[14] intersecting each of the four lines L_j, L_k, M_j, M_k. Thus we find 15 lines on top of the 12 ones in the double six, for a total of $15 + 12 = 27$, indicating that all 27 lines on a generic cubic play a role in the double six, either by being a true member, or by intersecting 2 pairs of lines.

Each pair of disjoint lines leads to a Schäfli double six, but 6 pairs lead to the same double six, so there is a total of $\frac{27\times16}{2\times6} = 36$ different possibilities.

Much remains to be said about the beautiful geometry and combinatorics of the 27 lines on a generic cubic surfaces. Let us simply quote that the incidence relations of the lines have a symmetry group which is isomorphic to the Weyl group of the exceptional simple Lie algebra E_6. This has natural explanations, and one of them popped out in "physics" recently. Quoting Wikipedia[15]:

[14] The line was called N when $L_j = L$, $M_j = M$ and $k = 1$.

[15] Again the author has all reasons to remain silent on these matters.

The 27 lines can be identified with the 27 possible charges of M-theory on a six-dimensional torus (6 momenta; 15 membranes; 6 fivebranes) and the group E_6 then naturally acts as the U-duality group. This is the so-called mysterious duality in M-theory.

In the following section, we return to a more old-fashioned subject.

2.9. Cubic surfaces are rational

One of the fundamental results is that cubic surfaces in $\mathbb{P}_3$ are rational.[16] More precisely we quote (see e.g. [11,13]):

Theorem 2. *Let $Q_1, \cdots, Q_6$ be six points in $\mathbb{P}_2$, no three collinear and not all laying on the same conic. There is a smooth cubic surface S in $\mathbb{P}_3$ and a polynomial map $f : S \mapsto \mathbb{P}_2$ such that $f^{-1}(P)$ is a singleton if $P \notin \{Q_1, \cdots, Q_6\}$ and a line on S if $P \in \{Q_1, \cdots, Q_6\}$, and moreover the lines are disjoint.*

One says that the six points $Q_1, \cdots, Q_6$ are blown-up to go from $\mathbb{P}_2$ to S. Note that five points in $\mathbb{P}_2$ always lay on a conic.

The reciprocal map can be described as follows. The homogeneous cubic polynomials in T, U, V vanishing at $Q_1 := (t_1; u_1; v_1), \cdots, Q_6 := (t_6; u_6; v_6)$ form a vector space of dimension 4, and if $P_1, \cdots, P_4$ is a basis for this vector space then $P_1, \cdots, P_4$ have no common zero except $Q_1, \cdots, Q_6$. Thus $g : \mathbb{P}_2 \setminus \{Q_1, \cdots, Q_6\} \mapsto \mathbb{P}_3(t; u; v) \to (P_1(t, u, v); P_2(t, u, v); P_3(t, u, v); P_4(t, u, v))$ is well defined and the image is a smooth cubic surface (with 6 disjoint lines removed).

To make contact with the discussion above, the 6 lines form the first half of a Schäfli double six.

In the following, we shall be more modest and show mainly on an example how a smooth cubic surface S can be seen as $\mathbb{P}_1 \times \mathbb{P}_1$ with 5 points blown up. This is a direct consequence of the fact that there are non-intersecting lines on a S, and exactly 5 lines on the surface join 2 given non-intersecting lines.

We start with an elementary observation. Let L, M be two non-intersecting lines in $\mathbb{P}_3$ and P a point neither on L nor on M. Then there is a single line N passing through P and meeting L and M. For instance we can assume, after a linear transformation, that $L : W = X = 0$ and $M : Y = Z = 0$. If $P = (w; x; y; z)$, the condition that P is not on L (resp. M) is that $(w, x) \neq (0, 0)$ (resp. $(y, z) \neq (0, 0)$) and one finds $N : xW - wX = zY - yZ = 0$, which intersects L at $P_L := (0; 0; y; z)$ and M at $P_M := (w; x; 0; 0)$.

Note that if P approaches a point of, say, L then in general anything may happen to the line N. However, if P approaches $P^* \in L$ while remaining on S, the line N has a limiting position. In fact, P_L approaches P^* and P_M approaches the intersection of M with the tangent plane to S at P^*, and N approaches the line defined by the two limiting points. Note that the tangent plane to S at P^* contains L so it cannot contain M because L and M do not intersect. The same argument works when P approaches a point of M. Thus we have a well-defined map $f : S \mapsto L \times M;\ P \to (P_L, P_M)$.

[16] Informally, this means that if S is a cubic surface there is a map from $\mathbb{P}_2$ (or $\mathbb{P}_1 \times \mathbb{P}_1$) to $\mathbb{P}_3$ whose components are given by polynomial expressions and whose image covers S exactly once except for some points forming a set of lower dimension (i.e. a finite number of curves) either in the source or in the target.

By construction, the line N cuts S at three distinct points if $P \notin L \cup M$ and at three points counting multiplicities if $P \in L \cup M$. Now S is given by a homogeneous cubic polynomial in 4 variables, and intersection with a line allows to eliminate 2, so the intersection is obtained by finding the zeros of a homogeneous cubic polynomial of two variables. Unless this polynomial is identically 0, it has exactly three zeros (counting multiplicities). The polynomial is identically 0 exactly when the line is contained in S. But there are exactly 5 lines on S intersecting both L and M, and these five lines do not intersect. Away from these 5 lines, $P \in S$ can be recovered from the knowledge of (P_L, P_M).

To be totally explicit, we do down-to-earth computations for the special case when the surface S is defined by

$$WY(W+Y) = XZ(X+Z).$$

It is a routine exercise to check that S is non-singular. In this form, 9 lines on S are obvious, and others are not difficult to spot. We shall meet some later, but we shall start with only $L : W = X = 0$ and $M : Y = Z = 0$.

If $P := (w; x; y; z) \in S$ is not on $L \cup M$ then $P_L = (0; 0; y; z)$ and $P_M = (w; x; 0; 0)$. Letting P approach L at $(0; 0; y; z)$ we find that P_L stays still at $(0; 0; y; z)$ while P_M approaches $(z^2; y^2; 0; 0)$. In the same way, if P approaches M at $(w; x; 0; 0)$ we find that P_M stays still at $(w; x; 0; 0)$ while P_L approaches $(0; 0; x^2; w^2)$. Viewing L as a copy of $\mathbb{P}_1$ with homogeneous coordinates $(y; z)$ and M as a copy of $\mathbb{P}_1$ with homogeneous coordinates $(w; x)$, we thus have a map

$$f : S \to \mathbb{P}_1 \times \mathbb{P}_1,$$
$$(w; x; y; z) \mapsto \begin{cases} ((w; x), (x^2; w^2)) & \text{if} \quad (y, z) = (0, 0) \\ ((z^2; y^2), (y; z)) & \text{if} \quad (w, x) = (0, 0) \\ ((w; x), (y; z)) & \text{else} \end{cases}.$$

Let $Q := (0; 0; y; z) \in L$ and $R := (w; x; 0; 0) \in M$, identified with $(y; z)$ and $(w; x)$ in the appropriate $\mathbb{P}_1$s. The line $[QR]$ has equations $xW - wX = zY - yZ = 0$, or to give an explicit correspondence with a $\mathbb{P}_1$, $[QR] = \{(uw; ux; vy; vz), (u; v) \in \mathbb{P}_1\}$. The images of $(0; 1)$ and $(1; 0))$ are on L and M respectively, and the equation for the intersection of N and S is

$$uvwy(uw + vy) = uvxz(ux + vz).$$

The solutions are $u = 0$ (the point Q on L) or $v = 0$ (the point R on M) or $u(w^2y - x^2z) = v(xz^2 - wy^2)$. The last equation has a single solution $(u; v) \in \mathbb{P}_1$ namely $(xz^2 - wy^2; w^2y - x^2z)$ unless $xz^2 - wy^2 = w^2y - x^2z = 0$.

Let K denote the set of solutions of

$$XZ^2 - WY^2 = W^2Y - X^2Z = 0$$

in $\mathbb{P}_1 \times \mathbb{P}_1$. They are easily found to be

$$((0; 1), (1; 0)),\ ((1; 0), (0; 1)),\ ((1; 1), (1; 1)),\ ((1; \xi), (1; \xi)),\ ((1; \xi^2), (1; \xi^2)),$$

where ξ is a primitive cube root of unity. Thus we find 5 special points as predicted by the general theory. Each point in $\mathbb{P}_1 \times \mathbb{P}_1$, in particular each point in K defines a line in $\mathbb{P}_3$, and the same equation $XZ^2 - WY^2 = W^2Y - X^2Z = 0$, this time in $\mathbb{P}_3$, has a set of solutions C which is made of 7 lines

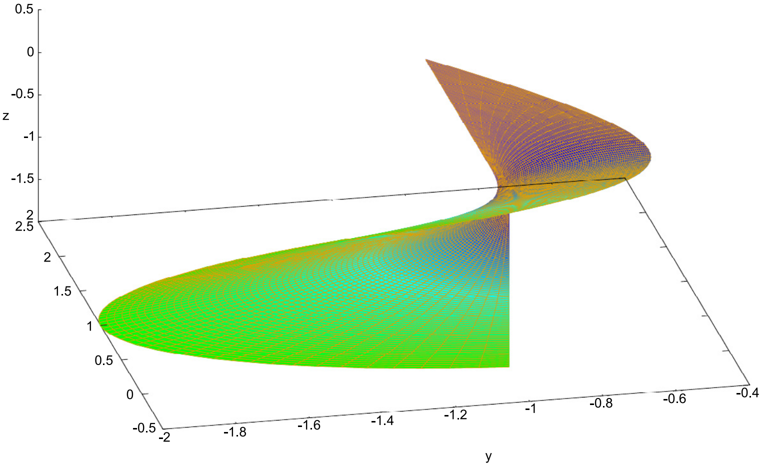

Fig. 1. A picture of the real section of the cubic surface $WY(W+Y) = XZ(X+Z)$.

$$W = X = 0 \quad Y = Z = 0 \quad X = Y = 0 \quad W = Z = 0$$
$$W - X = Y - Z = 0 \quad W - \xi X = Y - \xi Z = 0 \quad W - \xi^2 X = Y - \xi^2 Z = 0,$$

i.e. the lines L and M plus the 5 lines defined by the points in K, which are also, as expected, lines on S.

So we have a map

$$g : \mathbb{P}_1 \times \mathbb{P}_1 \setminus K \to S,$$
$$((w;x),(y;z)) \mapsto ((xz^2 - wy^2)w; (xz^2 - wy^2)x; (w^2y - x^2z)y; (w^2y - x^2z)z).$$

This map is given by polynomial formulae which are homogeneous – of degree 2 – separately in (w,x) and (y,z). Such a map is called rational.[17] The tangent plane to S at $(w;x;0;0) \in M$ has equation $w^2Y - x^2Z = 0$ and intersect L at $(0;0;x^2;w^2)$ and indeed $g((w;x),(x^2;w^2)) = (w;x;0;0)$ unless $((w;x),(x^2;w^2)) \in K$. The same argument shows that $g((z^2;y^2),(y;z)) = (0;0;y;z)$ unless $((z^2;y^2),(y;z)) \in K$. It is easily seen that $f \circ g = Id$. This shows that f is one-to-one, except for the five lines on S intersecting L and M, each of which being mapped to a point in K.

Thus S is "essentially" a copy of $\mathbb{P}_1 \times \mathbb{P}_1$, but with 5 points blown-up to 5 disjoint lines. An easy manipulation would show that S is also "essentially" a copy of $\mathbb{P}_2$ via the map

$$(t;u;v) \mapsto (u(v^2 - tu); t(v^2 - tu); t(u^2 - tv); v(u^2 - tv))$$

this time with 6 poins blown up.

The existence of an explicit rational formula makes it easy to draw the cubic surface, see Fig. 1.

[17] One of the explanation for choosing the name instead of polynomial is that in local coordinates, near points where, say, $x \neq 0$ in the first $\mathbb{P}_1$, $y \neq 0$ in the second $\mathbb{P}_1$, we find that the map is expressed via rational functions: $g((w;1),(1;z)) = (w; 1; \frac{w^2-z}{z^2-w}; \frac{w^2-z}{z^2-w}z)$.

Our last aim in this section is to explain briefly the term “blow-up”. For this, we make a local analysis at the points in K. We choose $Q := ((0; 1), (1; 0))$ for illustration.[18] If $((w; 1), (1; z)) \neq Q$, i.e. if $(w, z) \neq (0, 0)$,

$$g((w; 1), (1; z)) = ((w - z^2)w; w - z^2; z - w^2; (z - w^2)z).$$

What happens if $P := ((w; 1), (1; z))$ approaches Q along a certain direction, i.e. if $(w; z)$ remains fixed, but w and z go to 0? A convenient way to rephrase the question is to fix (w, z), set $P = ((\varepsilon w; 1), (1; \varepsilon z))$ and work to lowest order in ε. One gets

$$g((\varepsilon w; 1), (1; \varepsilon z)) = (0; w; z; 0) + 0(\varepsilon).$$

Thus, g maps points near Q to points near the line $f^{-1}(Q)$, and depending on the direction one approaches Q, g approaches a different point on $f^{-1}(Q)$.

Thus, we should interpret S as a copy of $\mathbb{P}_1 \times \mathbb{P}_1$, except that for each point $Q \in K$, S keeps track of the different directions one may approach Q. Thus the point Q is blown-up into a line parameterizing the different directions one may approach it.

There is an archetypal example of blow-up of a point: inside $\mathbb{P}_1 \times \mathbb{C}^2$ consider the set S of points $((w; z), (u, v))$ such that $uz = vw$. Thus S is a surface. An element of S is a line through the origin in $\mathbb{C}^2$ together with a point on that line. Of course, if the point is not the origin, it suffices to recover the line. But if the point is the origin, S keeps track of a direction. For $((w; z), (u, v)) \in S$ set $f((w; z), (u, v)) = (u, v) \in \mathbb{C}^2$, and for $(u, v) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ set $g(u, v) = ((u; v), (u, v)) \in S$. It is plain that $f \circ g = Id$. Thus S is $\mathbb{C}^2$ with the origin blown-up to keep track of a direction. Blow-ups can also be defined for objects more complicated than points. They are a crucial tool to remove singularities of algebraic objects. To give a trivial illustration, consider the curve $C : u^2 = v^2 - v^3$ in $\mathbb{C}^2$. It contains the point $(0, 0)$ and it is singular there: near $(0, 0)$ the curve has a double point with distinct tangents, and looks like the product of two lines $u \pm v = 0$, i.e. there are two ways to approach $(0, 0)$ along C. But $g(C \setminus \{(0, 0)\}))$ can be completed to a perfectly non-singular object inside S by adding the two points $((1; 1), (0, 0))$ and $((1; -1), (0, 0))$: the blow-up of $(0, 0)$ splits $(0, 0)$ on C in 2 points that keep track of the two distinct tangents.

This seems to be a natural place to stop this incursion in classical algebraic geometry.

3. Some invariant theory

The general theme of this section is on invariants associated to group actions: a group action of a group G on a set Σ is a map

$$G \times \Sigma \to \Sigma \quad (g, x) \mapsto g.x$$

with the conditions $e.x = x$ and $g.(h.x) = (gh).x$ for $g, h \in G$ and $x \in \Sigma$ (where e is the identity in G and gh stands for the product in G).

Two points $x, y \in \Sigma$ are equivalent if there is a $g \in G$ such that $y = g.x$. It is immediate from the conditions above that this is indeed an equivalence relation. The equivalence classes are called orbits, and the orbit $\{g.x,\ g \in G\}$ of $x \in \Sigma$ is denoted by either Gx or O_x. The set of orbits is denoted by Σ/G.

[18] The reader can check that the other special points lead to a similar interpretation.

It turns out that two types of (closely related) questions arise over and over when group actions occur: (i) Find a “canonical” representative in each orbit, or a section, i.e. a map $\varphi : \Sigma/G \to \Sigma$ such that $\varphi(O) \in O$ for each $O \in \Sigma/G$; (ii) Find a “natural” set F of functions on G that are invariant under G, i.e. constant on orbits (for each $f \in F$, $g \in G$ and $x \in \Sigma$, $f(g.x) = f(x)$) and separate the orbits (if $x, y \in \Sigma$ are on distinct orbits, there is an $f \in F$ such that $f(x) \neq f(y)$).

Very often, one has to be more modest and ask for a finite set of representatives in each orbit, or a sets of invariant functions which separate “most” orbits.[19]

Among the natural occurrences of such questions, the theoretical physicist[20] cannot help but think of gauge fixing in quantum field theory: in that case, Σ is the (infinite dimensional) affine space of connections, and G is the (infinite dimensional) gauge group. In the following, we shall meet an instance when Σ is a finite dimensional vector space $\mathbb{C}^{n+1}$ (or the associated projective space $\mathbb{P}_n$) and G is the group of linear transformations of determinant 1 acting on it.

3.1. (Linear) representations and their invariants

An action of a group G on a (say complex) vector space V (say over $\mathbb{C}$) by linear transformations is also called a (linear) representation of G on V. Such an action of G on V is equivalent to a group homomorphism ρ from G to $GL(V)$, the group of invertible linear transformations of V. We talk of the representation (V, ρ) of G and often write $\rho(g)v$ for $g.v$, $(g, v) \in G \times V$.

If (V, ρ) is a representation, the subspace V^G_ρ, or simply V^G when no confusion is possible, of vectors that are fixed by every element of G is a sub-representation.

Let (V, ρ) be finite dimensional representation of G. We define the action of G on V^* by $g.\mu :=^t g^{-1}(\mu)$, i.e. $\langle g.\mu, v\rangle = \langle \mu, g^{-1}.v\rangle$ for $v \in V$ and $\mu \in V^*$. As explained in Appendix A, this in turn induces an action of G on $S(V^*)$, the algebra of polynomial functions on V. Each $S_k(V^*)$, $k \geq 0$ is stable under the action of G.

Putting these notions together, we are lead to associate to each finite dimensional representation (V, ρ) of G the algebra $S(V^*)^G$ of G-invariant polynomial functions on V. Note that $S(V^*)^G$ is graded, and there is no ambiguity in the notation $S_k(V^*)^G$.

3.1.1. A structure theorem

The detailed structure of $S(V^*)^G$ can be quite involved, and computing it explicitly is usually hard. But under some restrictions on the group and/or the representation, some general features are known. Denoting by $SL_n(\mathbb{C})$, or simply SL_n the group of complex $n \times n$ matrices of determinant 1, one has (see e.g. [3]):

Theorem 3. *Suppose that $G = SL_n(\mathbb{C})$ and (V, ρ) is a finite dimensional representation of G. Then there are homogeneous elements $a_1, \cdots, a_n, b_1, \cdots, b_r \in S(V^*)^G$ such that the monomials $a_1^{i_1} \cdots a_n^{i_n} b_j$ with $i_1, \cdots, i_n \geq 0$ and $j = 1, \cdots, r$ form a (vector-space) basis of $S(V^*)^G$.*

19 This is because there are some restrictions on regularity either of the section or on the class of functions that one is willing to work with. This is also why we included vague notions as “canonical” or “natural” in the questions. It seems to the author that, without any restrictions, the axiom of choice gives a trivial positive answer.

20 Raymond could hardly have been an exception.

This result, quoted here in a down to earth fashion, holds for much more general groups[21]. A weaker form of this result, namely that $S(V^*)^G$ can be generated by the monomials in a finite number of homogeneous elements, was proven by Hilbert in the case when $G = SL_n(\mathbb{C})$, and is at the origin of all the more recent developments.

The algebra structure of $S(V^*)^G$ then implies the existence and uniqueness of polynomials $P_{jj'}^{j''} \in \mathbb{C}[X_1, \cdots, X_n]$ such that

$$b_j b_{j'} = \sum_{j''=1}^{r} P_{jj'}^{j''}(a_1, \cdots, a_n) b_{j''}.$$

The polynomials $P_{jj'}^{j''}$ satisfy quadratic relations that embody the associativity of $S(V^*)^G$. Commutativity is just the symmetry of $P_{jj'}^{j''}$ under the exchange of j and j'. These relations in turn characterize fully the algebra structure of $S(V^*)^G$.

Note that $S_0(V^*)^G$ is always invariant, and it has dimension 1. All other invariants have positive degree, so exactly one of $b_1, \cdots, b_r$, say b_{j_0} has degree 0, all other b_js and all a_ls have positive degree.

For fixed j, consider the matrix M whose matrix elements are the $P_{jj'}^{j''}$, $j', j'' = 1, \cdots, r$, i.e. polynomials in $a_1, \cdots, a_n$. If $C(Y, X_1, \cdots, X_n)$ is any polynomial, we infer that

$$C(b_j, a_1, \cdots, a_n) b_{j'} = \sum_{j''=1}^{r} C(M, a_1, \cdots, a_n)_{j'}^{j''} b_{j''}.$$

Now specialize to $C(Y) := \mathrm{Det}\,(M - Y)$, the characteristic polynomial of M, a polynomial in Y but also in $a_1, \cdots, a_n$. By the Cayley–Hamilton theorem, $C(M) = 0$ as a matrix. Then $C(b_j)b_{j'} = 0$ for each j', in particular $j' = j_0$. Thus $C(b_j) = 0$. This means that each b_j satisfies an algebraic equation of degree at most r whose coefficients are polynomials in $a_1, \cdots, a_n$.

3.1.2. Molien series

The Molien series of (V, ρ) is the formal series

$$F_{V,\rho}(t) = \sum_k t^k \dim S_k(V^*)^G$$

There are several techniques to compute the first terms in the t-expansion of $F_{V,\rho}(t)$. It happens quite often that $F_{V,\rho}(t)$ makes sense as a holomorphic function in the neighborhood of 0, or even better as a rational function. The previous proposition implies this when $G = SL_n(\mathbb{C})$. Indeed,

Proposition 9. *Let (V, ρ) be a finite dimensional representation of $G = SL_n(\mathbb{C})$. Let $a_1, \cdots, a_n$, $b_1, \cdots, b_r$ be as in Proposition 3. Let δ_l be the degree of a_l, $l = 1, \cdots, n$ and Δ_j be the degree of b_j, $j = 1, \cdots, r$. Then*

$$F_{V,\rho}(t) = \frac{\sum_{j=1}^{r} t^{\Delta_j}}{\prod_{l=1}^{n}(1 - t^{\delta_l})}.$$

[21] It holds for instance for finite groups, or for Lie subgroups of any $SL_n(\mathbb{C})$. For an even larger class of groups it holds with some mild restriction on the representation.

Let us note that using Theorem 3 to prove Proposition 9 is a bit like using a sledgehammer to crack a walnut.

The proof is elementary: the monomial $a_1^{i_1} \cdots a_n^{i_n} b_j$ contributes in $S_k(V^*)^G$ for $k = \Delta_j + \sum_{l=1}^n i_l \delta_l$. So

$$F_{V,\rho}(t) = \sum_{j=1}^{r} \sum_{i_1, \cdots, i_n \geq 0} t^{\Delta_j + \sum_{l=1}^n i_l \delta_l}.$$

Thus the problem is reduced to the summation of geometric series, which leads immediately to the announced result.

Of course, the $n + r$-tuples $a_1, \cdots, a_n, b_1, \cdots, b_r$ doing the job in Theorem 3 are far from unique (why ?). However, the Molien series gives a number of constraints. In particular, $F_{V,\rho}(t)$ has a pole of order n at $t = 1$, and the coefficient is $\frac{r}{\prod_{l=1}^n \delta_l}$. Thus n is well-defined, and deserves to be called the dimension of $S(V^*)^G$: informally it describes the algebra of functions on a space that needs n independent coordinates to specify a point. As noticed above, given the a_ls, the b_js just lead to a finite degeneracy.

Informally again, the space whose algebra of functions is $S(V^*)^G$ should be the space of orbits. But it turns out that this question, which is at the heart of Mumford's geometric invariant theory, is very subtle. We shall content with naïve remarks.

3.1.3. *Separation of orbits by invariants*

Though our main interest is in $G = SL_n(\mathbb{C})$, we shall only prove a result that holds for finite groups:

Proposition 10. *Let (V, ρ) be a finite dimensional representation of a finite group G. The invariants in $S(V^*)^G$ separate the G-orbits in V.*

We start with the remark that V^* separates points in V: if $v_1, \cdots, v_m$ are distinct vectors in V there is a linear form $\mu \in V*$ such that the scalars $\langle \mu, v_1 \rangle, \cdots, \langle \mu, v_m \rangle$ are all distinct. The proof goes by recursion on m. If $m = 1$ there is nothing to prove. If $m = 2$, $v_2 - v_1 \neq 0$ implies that for some $\nu \in V^*$, $\langle \nu, v_2 - v_1 \rangle \neq 0$. Suppose the result is proven for $m \leq M$ where $M \geq 2$. Let $v_1, \cdots, v_{M+1}$ be distinct vectors in V. By the induction hypothesis, there is a $\mu \in V^*$ such that $\langle \mu, v_1 \rangle, \cdots, \langle \mu, v_M \rangle$ are all distinct. If $\langle \mu, v_{M+1} \rangle$ does not appear in the previous list, we are done. Else, we may assume that $\langle \mu, v_{M+1} \rangle = \langle \mu, v_M \rangle$. Choose $\nu \in V^*$ such that $\langle \nu, v_{M+1} \rangle \neq \langle \nu, v_M \rangle$. For $\varepsilon \in \mathbb{C}$ define $\mu_\varepsilon := \mu + \varepsilon \nu$. For ε small enough, we still have that the $\langle \mu_\varepsilon, v_1 \rangle, \cdots, \langle \mu_\varepsilon, v_M \rangle$ are all distinct. Thus if ε is non-zero but small enough, the $\langle \mu_\varepsilon, v_1 \rangle, \cdots, \langle \mu_\varepsilon, v_{M+1} \rangle$ are all distinct.

We can now use standard Lagrange interpolation: if μ separates $v_1, \cdots, v_m$ ($m \geq 1$) and $v \in V$, define, for $l = 1, \cdots, m$

$$P_l(v) := \prod_{k \neq l} \frac{\langle \mu, v - v_k \rangle}{\langle \mu, v_l - v_k \rangle}.$$

Of course $P_l \in S_{m-1}(V^*)$, and $P_l(v_k) = \delta_{kl}$.

Now suppose that $v_1, \cdots, v_m$ enumerates a disjoint union of orbits of the action of the finite group G on V. That is, suppose that $\{1, \cdots, m\}$ can be partitioned as $I_1 \cup \cdots \cup I_n$ in such a way that the v_ls, $l \in I_k$, form an orbit O_k.

If $l \in I_k$, we have

$$P_l(v) = \prod_{l' \in I_k \setminus \{l\}} \frac{\langle \mu, v - v_{l'} \rangle}{\langle \mu, v_l - v_{l'} \rangle} \prod_{l'' \notin I_k} \frac{\langle \mu, v - v_{l''} \rangle}{\langle \mu, v_l - v_{l''} \rangle}.$$

The second product involves entire orbits, and remembering that the order of the subgroup of G fixing v_l is $\frac{\#G}{\#O_k}$, one checks that

$$\frac{\#O_k}{\#G} \sum_{g \in G} P_l(g.v)$$

is an invariant polynomial that depends on l only via the orbit O_k of v_l and takes value 1 for $v \in O_k$ but 0 for $v \in O_{k'}$, $k' \neq k$. Thus polynomial invariants separate the orbits: given a finite number of orbits, there is a polynomial invariant that take value 1 on a prescribed orbit and 0 on the others. This finishes the proof.

The use of an averaging procedure, which presents no analytic difficulty when G is finite and appears in various disguises for more general groups, is typical.

The question whether such a strong result holds for more general group has a negative answer. The reason is simple: in general orbits do not need to be closed but polynomials are continuous functions, which if constant on an orbit must be constant on its closure (which is obviously a union of orbits). But this is the only difficulty and we have:

Proposition 11. *Let (V, ρ) be a finite dimensional representation of $G = SL_n(\mathbb{C})$. The invariants in $S(V^*)^G$ separate the closed G-orbits in V.*

Let us note that this result holds for much more general groups, and that the most natural topology to study these questions is not the usual topology but the Zariski topology of algebraic geometry.

We now turn to an elementary example illustrating the previous discussions.

3.2. Warm-up with cubics in $\mathbb{P}_1$: SL_2

In the subsection we give in great details the invariant theory for the action of SL_2 on the space $\Sigma_{2,3}$ of (non-trivial) homogeneous polynomials of degree 3 in 2 variables. This is really only an exercise, but one via which many lessons can be learned. Our real goal, the study of invariants for the action of SL_4 on the space $\Sigma_{4,3}$ of (non-trivial) homogeneous polynomials of degree 3 in 4 variables is a much more difficult enterprise, but all the basic ingredients have their source in the methods provided in this section.

So write a general element of $\Sigma_{2,3}$ as

$$P(u, v) := c_{3,0}u^3 + c_{2,1}u^2v + c_{1,2}uv^2 + c_{0,3}v^3.$$

The zero locus $P(u, v) = 0$ in $\mathbb{P}_1$ is made of 3 points, possibly not all distinct. Conversely, given 3 points in $\mathbb{P}_1$ with representatives (u_0, v_0), (u_1, v_1) and (u_∞, v_∞) in $\mathbb{C}^2 \setminus (0, 0)$, the homogeneous polynomial of degree 3

$$(v_0u - u_0v)(v_1u - u_1v)(v_\infty u - u_\infty v)$$

has exactly the 3 points (counted with multiplicity) as its zero locus. This configuration is non-singular when the 3 points are distinct, and singular otherwise.[22]

The group SL_2 acts on $\mathbb{P}_1$, hence on $\Sigma_{2,3}$. The explicit formulae are the following: the matrix $g = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, $ps - qr = 1$ acts on $\begin{pmatrix} u \\ v \end{pmatrix}$ as

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

According to our general rules, we define $P' = g.P$ by $P'(u, v) = P(u', v')$ where $\begin{pmatrix} u' \\ v' \end{pmatrix}$ is obtained by acting on $\begin{pmatrix} u \\ v \end{pmatrix}$ with g^{-1} i.e. $P'(u, v) = P(su - qv, -ru + pv)$, leading to

$$\begin{pmatrix} c_{3,0} \\ c_{2,1} \\ c_{1,2} \\ c_{0,3} \end{pmatrix} \to \begin{pmatrix} c'_{3,0} \\ c'_{2,1} \\ c'_{1,2} \\ c'_{0,3} \end{pmatrix} = \begin{pmatrix} s^3 & -rs^2 & r^2 s & -r^3 \\ -3qs^2 & ps^2 + 2qrs & -2prs - qr^2 & 3pr^2 \\ 3q^2 s & -2pqs - q^2 r & p^2 s + 2pqr & -3p^2 r \\ -q^3 & pq^2 & -p^2 q & p^3 \end{pmatrix} \begin{pmatrix} c_{3,0} \\ c_{2,1} \\ c_{1,2} \\ c_{0,3} \end{pmatrix}.$$

3.2.1. Main results

The result on invariants is:

Proposition 12. *The ring of polynomial invariants is generated by one invariant I_4, of degree* 4

$$I_4 = -27\, c_{0,3}^2\, c_{3,0}^2 + 18\, c_{0,3}\, c_{1,2}\, c_{2,1}\, c_{3,0} - 4\, c_{1,2}^3\, c_{3,0} - 4\, c_{0,3}\, c_{2,1}^3 + c_{1,2}^2\, c_{2,1}^2.$$

The result on orbits is:

Proposition 13. *The invariant I_4 is non-zero on non-singular orbits. Those are closed and separated by I_4. The zero locus of I_4 is made of the singular orbits and consists of two orbits, the orbit of configurations with one simple point and one double point, and the orbit of configurations of triple points. The first is not closed but its closure contains the second, which is closed.*

This illustrates nicely the problem of the separation of orbits by invariants.

Both propositions will be proved below.

3.2.2. Brute force search for invariants

The action of diagonal matrices $g = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$ is easiest to analyze. It leads to

$$\begin{pmatrix} c_{3,0} \\ c_{2,1} \\ c_{1,2} \\ c_{0,3} \end{pmatrix} \to \begin{pmatrix} c'_{3,0} \\ c'_{2,1} \\ c'_{1,2} \\ c'_{0,3} \end{pmatrix} = \begin{pmatrix} p^{-3} c_{3,0} \\ p^{-1} c_{2,1} \\ p c_{1,2} \\ p^3 c_{0,3} \end{pmatrix}.$$

[22] This can be taken as a mere convenient definition but is also the one coming from computation of the Jacobian.

The action of triangular matrices $g = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$, is also quite simple, leading to

$$\begin{pmatrix} c_{3,0} \\ c_{2,1} \\ c_{1,2} \\ c_{0,3} \end{pmatrix} \to \begin{pmatrix} c'_{3,0} \\ c'_{2,1} \\ c'_{1,2} \\ c'_{0,3} \end{pmatrix} = \begin{pmatrix} c_{3,0} \\ c_{2,1} - 3qc_{3,0} \\ c_{1,2} - 2qc_{2,1} + 3q^2c_{3,0} \\ c_{0,3} - qc_{1,2} + q^2c_{2,1} - q^3c_{3,0} \end{pmatrix}$$

and

$$\begin{pmatrix} c_{3,0} \\ c_{2,1} \\ c_{1,2} \\ c_{0,3} \end{pmatrix} \to \begin{pmatrix} c'_{3,0} \\ c'_{2,1} \\ c'_{1,2} \\ c'_{0,3} \end{pmatrix} = \begin{pmatrix} c_{3,0} - rc_{2,1} + r^2c_{1,2} - r^3c_{0,3} \\ c_{2,1} - 2rc_{1,2} + 3r^2c_{0,3} \\ c_{1,2} - 3rc_{0,3} \\ c_{0,3} \end{pmatrix}.$$

In fact any transformation in SL_2 can be obtained by an appropriate succession of the above three kinds of transformations, and as SL_2 is connected, infinitesimal transformations are enough. So a polynomial (or any smooth function in fact) in $c_{3,0}$, $c_{2,1}$, $c_{1,2}$, $c_{0,3}$ is invariant if and only if it is annihilated by the three differential operators:

$$3c_{3,0}\frac{\partial}{\partial c_{3,0}} + c_{2,1}\frac{\partial}{\partial c_{2,1}} - c_{1,2}\frac{\partial}{\partial c_{1,2}} - 3c_{0,3}\frac{\partial}{\partial c_{0,3}}$$
$$3c_{3,0}\frac{\partial}{\partial c_{2,1}} + 2c_{2,1}\frac{\partial}{\partial c_{1,2}} + c_{1,2}\frac{\partial}{\partial c_{0,3}}$$
$$c_{2,1}\frac{\partial}{\partial c_{3,0}} + 2c_{1,2}\frac{\partial}{\partial c_{2,1}} + 3c_{0,3}\frac{\partial}{\partial c_{1,2}}$$

The three types of matrices (diagonal, upper-triangular, lower-triangular) are subgroups of SL_2, and the infinitesimal generators are

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with the familiar commutation relations

$$[E_+, E_-] = H,\ [H, E_\pm] = \pm 2E_\pm.$$

Take $g \in SL_2$ such that the adjoint action of g fixes diagonal matrices globally. Such gs can be composed, and they induce invertible linear transformations on the space of diagonal matrices, leading to a group W of linear transformations acting on diagonal matrices. One checks explicitly that W has order 2. One element of SL_2 inducing a generator of W is $g := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for whom one checks that:

$$gHg^{-1} = -H \quad gE_\pm g^{-1} = -E_\mp.$$

The group W is called the Weyl group of SL_2.

What can be seen simply via the action of diagonal matrices is that invariants involve only the combinations $w := c_{0,3}c_{3,0}$, $x := c_{1,2}c_{2,1}$, $y := c_{1,2}^3c_{3,0}$ and $z := c_{0,3}c_{2,1}^3$. Acting on functions of w, x, y, z, the differential operators corresponding to triangular matrices become

$$y\frac{\partial}{\partial w} + (3y + 2x^2)\frac{\partial}{\partial x} + 6xy\frac{\partial}{\partial y} + (9wx^2 + x^3)\frac{\partial}{\partial w}$$
$$z\frac{\partial}{\partial w} + (3z + 2x^2)\frac{\partial}{\partial x} + 6xz\frac{\partial}{\partial z} + (9wx^2 + x^3)\frac{\partial}{\partial w}$$

This is to be used to check that $I_4 = -27w^2 + 18wx - 4y - 4z + x^2$ is indeed an invariant (do it!) but even for such a simple case it seems hard to get a hold on the structure of the algebra of polynomial invariants by this procedure.

3.2.3. The discriminant

The action of SL_2 on triples of points in $\mathbb{P}_1$ preserves the "geometry". In particular, it preserves the fact that the configuration is singular or not, i.e. that the 3 points are distinct or not, i.e. that P has only simple roots or not. To $P(u,v)$ we associate the polynomial in 1 variable $p(z) := P(z,1)$. This polynomial has less than 3 distinct zeros if and only if the determinant:

$$\begin{vmatrix} c_{0,3} & c_{1,2} & c_{2,1} & c_{3,0} & 0 \\ 0 & c_{0,3} & c_{1,2} & c_{2,1} & c_{3,0} \\ c_{1,2} & 2c_{2,1} & 3c_{3,0} & 0 & 0 \\ 0 & c_{1,2} & 2c_{2,1} & 3c_{3,0} & 0 \\ 0 & 0 & c_{1,2} & 2c_{2,1} & 3c_{3,0} \end{vmatrix}$$

vanishes.[23] The determinant turns out to be $-c_{3,0}I_4$. The spurious factor $c_{3,0}$ comes from the fact that if $c_{3,0} = 0$ one of the points is at ∞ and is not seen by $p(z)$. Thus we conclude that the vanishing of I_4 is the condition for singularity of the configuration. Then I_4 *must* be an invariant: all that could happen is that it would be multiplied by a global factor under SL_2 transformations, but SL_2 has no non-trivial representation of dimension 1.

For those unfamiliar with elimination theory, there is a more direct access to the condition for singularity: factoring $P(u,v)$ as $(v_0u - u_0v)(v_1u - u_1v)(v_\infty u - u_\infty v)$, the configuration is singular if and only if

$$D := (u_0v_1 - v_0u_1)(u_1v_\infty - v_1u_\infty)(u_\infty v_0 - v_\infty u_0) = 0.$$

The polynomial D, called the discriminant, is antisymmetric under the exchange of two roots, but its square is symmetric and by elementary results it has to be expressible as a polynomial in $c_{0,3}, c_{1,2}, c_{2,1}, c_{3,0}$. These coefficients are homogeneous of degree 3 in the u_is and v_is and D^2 is homogeneous of degree 12. Hence D^2 is homogeneous of degree 4 in $c_{0,3}, c_{1,2}, c_{2,1}, c_{3,0}$. Doing the algebra, one checks without great surprise that

$$I_4 = D^2.$$

This simple example points to one of the powerful tools to define invariants: find a geometric (i.e. invariant under symmetry transformations) feature that is non-generic but appears in codimension one. Here, the generic situation is when the 3 points are distinct, and by adjusting one parameter one forces 2 of the 3 points to coincide. There is an associated invariant polynomial.

In more complicated situations, powerful techniques are available to find, and sometimes exhaust, invariants. But we shall not follow this route.

3.2.4. "Canonical" representatives

It is well known that linear fractional transformations acting on $\mathbb{P}_1$ act transitively on triplets of distinct points, that is, there is a single linear fractional transformation sending an arbitrary triplet of distinct points (z_0, z_1, z_∞) to $(0, 1, \infty)$. The usual argument can be adapted to our situation. We write again $P(u,v) = (v_0u - u_0v)(v_1u - u_1v)(v_\infty u - u_\infty v)$ and look for a $g \in SL_2$

[23] This in fact a byproduct of elimination theory to be recalled below (see Section 3.3).

such that g maps $(u_0; v_0)$ to $(0; 1)$, $(u_1; v_1)$ to $(1; 1)$ and $(u_\infty; v_\infty)$ to $(1; 0)$. Setting $D_0 := (v_1 u_\infty - u_1 v_\infty)$, $D_1 := (v_\infty u_0 - u_\infty v_0)$ and $D_\infty := (v_0 u_1 - u_0 v_1)$ so that $D = D_0 D_1 D_\infty$ one finds after elementary algebraic manipulations that:

$$g = D^{-1/2} \begin{pmatrix} D_0 & 0 \\ 0 & D_\infty \end{pmatrix} \begin{pmatrix} v_0 & -u_0 \\ -v_\infty & u_\infty \end{pmatrix}.$$

The inverse is

$$g^{-1} = D^{-1/2} \begin{pmatrix} u_\infty & u_0 \\ v_\infty & v_0 \end{pmatrix} \begin{pmatrix} D_\infty & 0 \\ 0 & D_0 \end{pmatrix}$$

Thus $gP(u, v) = D^{-1/2} uv(u - v)$. In the above computation, we note that there are in fact two choices, for the square root $D^{-1/2}$, but the same choice has been kept all along.

Also, the choice of sign of D itself changes under permutation of the three points: D is assigned to an ordered triple, but the zero locus of a non-singular P is an unordered triple of points. Thus, we have shown:

Proposition 14. *The orbit under SL_2 of a non-singular P with invariant I_4 contains* 4 *special representatives of the form*

$$I_4^{1/4} uv(u - v)$$

corresponding to the four possible roots $I_4^{1/4}$.

To complete this analysis, we start from a polynomial P in the canonical form $P = cuv(u - v)$ and note that[24] the corresponding SL_2 invariant is c^4. It remains to examine the subgroup of SL_2 fixing the canonical form, i.e. fixing (globally) the set $\{(0; 1), (1; 1), (1; 0)\}$. This subgroup has order 12, because each permutation (there are 6 of them) can be implemented by two elements of SL_2 (differing by a sign). But only a quotient of order 4 acts on P, and it is generated by $g = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ which take $c \to -ic$. Any polynomial invariant is thus a polynomial in c^4. This leads to the now obvious result (already quoted above but which we repeat here for convenience):

Proposition 15. *The algebra of polynomial invariants for the action of SL_2 on the space $\Sigma_{2,3}$ of polynomials of degree* 3 *in* 2 *variables is a polynomial algebra with one generator of degree* 4*,*

$$I_4 = -27\, c_{0,3}^2\, c_{3,0}^2 + 18\, c_{0,3}\, c_{1,2}\, c_{2,1}\, c_{3,0} - 4\, c_{1,2}^3\, c_{3,0} - 4\, c_{0,3}\, c_{2,1}^3 + c_{1,2}^2\, c_{2,1}^2.$$

Canonical representatives are another powerful tool to understand the structure of invariants. The contrast between the general formula for I_4, which is non-trivial even for such an elementary example, and the formula for I_4 for canonical representatives is a strong hint that they lead to much more manageable explicit computations. They allow to reduce the action of a continuous group to the action of a finite group, for whom the study of invariants proves to be much simpler. This is the route followed by Salmon for cubic surfaces [19].

[24] Obviously from the proposition above, but also by plugging the explicit coefficients of this P, $c_{3,0} = c_{0,3} = 0$, $c_{2,1} = -c_{1,2} = c$ in the formula for I_4.

3.2.5. Characters and Molien series

Another mechanical way to get a grasp at invariants, i.e. to at least count them, is via Molien series[25]. This is done via the following trick. As we observed in Section 3.2.2, the action of diagonal matrices $g = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$ breaks $\Sigma_{2,3}$ as a direct sum of 4 1-dimensional subspaces on which g acts as p^{-3}, p^{-1}, p and p^3. Thus $\mathrm{Tr}_{\Sigma_{2,3}}\, g = p^{-3} + p^{-1} + p + p^3$. We introduce a variable z and rewrite this formally as $\chi(\Sigma_{2,3}) = z^3 + z + z^{-1} + z^{-3}$.

The symbol χ is called a character, and it is a bookkeeping device for the eigenvalues of the action of diagonal gs on a representation. We take the trace of $g = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$ on the representation, and replace p by z.

Note that if $V = \mathbb{C}^2$ with the action of $G = SL_2(\mathbb{C})$ we have that $\Sigma_{2,3} = S_3(V^*)$. The algebra of polynomials on $\Sigma_{2,3}$ is thus $S(S_3(V^*)^*)$ and we are looking for $S(S_3(V^*)^*)^G$. Clearly $\chi(V) = z + z^{-1}$ so by the formulae recalled in Appendix A we find $\chi(V^*) = z + z^{-1}$, and $\frac{1}{(1-tz)(1-tz^{-1})} = \sum_k t^k \chi(S_k(V^*))$. We retrieve $\chi(\Sigma_{2,3}) = z^3 + z + z^{-1} + z^{-3}$ and $\chi(\Sigma^*_{2,3}) = z^3 + z + z^{-1} + z^{-3}$. The fact that χ is the same for a space and its dual is very special to SL_2, a fact to be kept in mind later when we turn to SL_4 even if it has no impact on the counting of invariants.[26] Going one step further, we have

$$\frac{1}{(1-tz^3)(1-tz)(1-tz^{-1})(1-tz^{-3})} = \sum_k t^k \chi(S_k(S_3(V^*)^*)).$$

Thus, we know in principle the eigenvalues for the action of a diagonal g on $S_k(S_3(V^*)^*)$. The question that remains now is whether or not this allows to extract the number of times $S_k(S_3(V^*)^*)$ contains the trivial representation and[27] how this multiplicity can be computed in practice?

Before attacking this question, let us rephrase it in terms more familiar to physicists. By restricting $SL_2(\mathbb{C})$ to the subgroup of unitary matrices, we can view V as the spin $1/2$ representation, and $\Sigma_{2,3} = S_3(V^*)$ as the spin $3/2$ representation, obtained as the symmetric piece in the space of 3 spins $1/2$. Thus the question we are addressing is to count the number of singlet states that can be built by symmetric wave functions of k spins $3/2$. The familiar commutation relations recovered in Section 3.2.2, namely

$$[E_+, E_-] = H, \; [H, E_\pm] = \pm 2E_\pm,$$

can be used to check without much trouble that the finite dimensional representations of SL_2 are completely reducible and that there is exactly one irreducible representation V_k, of dimension k for each $k = 1, 2, \cdots$[28] Of course V_1 is the trivial representation and V_2 describes spin $1/2$. It is easy to check that

[25] For a general overview of the use of generating functions in counting problems, including a discussion of the information than can be extrated from the singularity structure,see the wonderful [7].

[26] In that a representation on a space and on its dual have the same number of invariants.

[27] If the answer is yes. It is yes!

[28] Physicists know from their first introduction to Quantum Mechanics that this statement holds when SL_2 is replaced by the subgroup SU_2 of unitary matrices and "finite dimensional" is replaced by "unitary", but a direct argument for SL_2 goes along the same lines.

$$\chi(V_k) = z^{k-1} + z^{k-3} + \cdots + z^{-k+1} = \frac{z^k - z^{-k}}{z - z^{-1}},$$

conveying that the eigenvalue of H on V_k are $k-1, k-3, \cdots, -k+1$.

The character of V_1 is $z^0 = 1$, but the monomial z^0 appears in the character for any odd value of k. However, for k odd and > 1, the term 1 always appears in V_k in company of a z^{-2} and of a z^2. Thus, to count the number of invariants in a finite dimensional representation W of SL_2 it suffices to subtract the coefficient of z^{-2} or of z^2 from the coefficient of z^0. We could also subtract the sum of the coefficients of z^2 and z^{-2} from twice the coefficient of z^0 and divide by 2.

Thus indeed, the knowledge of the character of W is enough to recover the number of invariants in W. We can turn the above three counting recipes into "explicit" formulae using contour integrals[29]:

$$\dim W^G = \oint \frac{dz}{z}(1-z^2)\chi(W) = \oint \frac{dz}{z}(1-z^{-2})\chi(W) = \oint \frac{dz}{z}\frac{(z-z^{-1})^2}{-2}\chi(W).$$

It is clear that the first two formulae are essentially the same. They are a bit less symmetric but a bit simpler than the last one, which somehow involves counting everything twice. As a side remark, we note that the factor $1/2$ in the last formula has a natural interpretation: the denominator 2 is the order of the Weyl group of SL_2. Later, when we generalize to the invariants for the action of SL_4 on cubic surfaces, we shall have to face the fact that the Weyl group of SL_4 has order $4! = 24$ so having a formula avoiding multiple counting is good news. In fact, the last formula is closely related, via analytic continuation, to the general theory of orthogonality of characters, the Weyl formula, etc. (see B.6). Just note that it can be rewritten as

$$\dim W^G = \oint \frac{dz}{z}\frac{-1}{2}\Big[(z-z^{-1})\chi(W)\Big]\Big[(z-z^{-1})\chi(V_1)\Big].$$

Now for $z = e^{i\theta}$ we observe that $(z-z^{-1})\chi(V_k) = 2i\sin k\theta$ and, specializing the contour to $|z| = 1$

$$\oint \frac{dz}{z}\frac{(z-z^{-1})^2}{-2}\chi(V_k)\chi(V_l) = \int_0^{2\pi}\frac{d\theta}{\pi}\sin k\theta\sin l\theta = \delta_{k,l}.$$

This is to be compared with the probably more standard formula

$$\oint_{|z|=1} \frac{dz}{z}\frac{|(z-z^{-1})^2|}{2}\chi(V_k)\overline{\chi(V_l)} = \delta_{k,l}.$$

Going back to our main interest, we thus have to evaluate, say using the second formula,

$$\sum_k t^k \dim(S_k(S_3(V^*)^*))^G = \oint \frac{dz}{z}(1-z^{-2})\frac{1}{(1-tz^3)(1-tz)(1-tz^{-1})(1-tz^{-3})}.$$

The last thing we have to care of is the integration contour. If the left-hand side is expanded formally in powers of t taken as a formal variable before the z-integration is done, the contour is immaterial. However, if t is taken as a complex number, the expansion in powers of t at small

29 We use the convention that the symbol $\oint$ includes the omnipresent $\frac{1}{2i\pi}$.

t has to be exchanged with the z-integral, a simple instance of Fubini's theorem. A moment thinking shows that if the contour of integration is $|z| = 1$ the inversion of the small t expansion and the z integration can be permuted.

The integral can be computed by contour deformation in terms of residues and only the poles inside the unit disc do contribute, i.e. the poles at $z = t$ and $z^3 = t$. Let us anticipate that the result is, as expected from the other approaches, that

$$\sum_k t^k \dim(S_k(S_3(V^*)^*))^G = \frac{1}{1-t^4}$$

which says precisely that there is exactly one invariant (modulo scaling) when the degree is a multiple of 4, and no invariant in the other cases.

We now devote some time to the efficient computation of residues. We shall then compute the above integral as a simple application.

3.3. Intermezzo: efficient computation of residues

In the last section, we encountered integrals of the type:

$$I = \oint_C \frac{dz}{R(z)} \frac{P(z)}{Q(z)}$$

where $P(z)$, $Q(z)$, $R(z)$ are polynomials, $Q(z)$ and $R(z)$ having no common zeros, and C is a contour with index 1 at each zero of R but encircling no zero of Q.

Of course, the answer is given by Cauchy's theorem. If $R(z)$ factorizes as $R(z) = r\prod_\alpha (z - z_\alpha)^{n_\alpha}$ we get, setting $R_\alpha(z) := r\prod_{\beta\neq\alpha}(z - z_\beta)^{n_\beta}$:

$$I = \sum_\alpha \frac{1}{(n_\alpha - 1)!} \left(\frac{d^{n_\alpha - 1}}{dz^{n_\alpha - 1}} \frac{P(z)}{Q(z)R_\alpha(z)} \right)_{|z = z_\alpha}.$$

However, this closed formula hides one important property: despite the explicit appearance of the roots of $R(z)$, I is in fact a rational function of the coefficients of $P(z)$, $Q(z)$ and $R(z)$. The alternative formula we shall give in a moment for I makes it obvious. Many other formal proofs can be built, but we should prefer those that can be efficiently implemented on a computer.

To see the kind of difficulties, let us deal with the simplest case, when all the multiplicities n_α are equal to 1, i.e. when $R(z)$ has simple zeros. Then

$$I = \sum_\alpha \frac{P(z_\alpha)}{Q(z_\alpha)R'(z_\alpha)}$$

This is a symmetric function of the roots of R so by Newton's theorem it can be expressed solely in terms of the coefficients of $P(z)$, $Q(z)$ and $R(z)$. But how can one make it explicit? Experiences with the computer show that though formal algebra packages can implement versions of Newton's algorithm or play other tricks, the amount of computation becomes prohibitive when the polynomials are complicated. It seems that one of the reasons is that exploitation of the symmetric functions of the roots occurs only after the above sum of fractions has been reduced to the same denominator, so that the numerator and denominator are symmetric polynomials. Imagine that Q and R have degree of order n. Then the denominator involves about n^n terms. So even rather modest values of n need a large memory and a large number of simplification steps.

The method we finally came up with needs much less: it involves the solution of a linear system in n unknowns (which of course should not be solved by the general Cramer's formulae).

3.3.1. Generalities

We start with the following elementary proposition, which is the basis of elimination theory:

Proposition 16. *Two polynomials $U(z)$ and $V(z)$ of degree m and n respectively have a nontrivial common factor if and only if there are nonzero polynomials $v(z)$ and $u(z)$ of degree at most $n-1$ and $m-1$ such that $v(z)U(z) = u(z)V(z)$.*

We follow closely [12]. One direction is obvious: if $w(z)$ is a nontrivial divisor of both $U(z)$ and $V(z)$, write $U(z) = w(z)u(z)$ and $V(z) = w(z)v(z)$, so that degree of $u(z)$ is $< m$, the degree of $v(z)$ is $< n$ and $v(z)U(z) = u(z)V(z)$. This uses only the fact that the product of two nonzero polynomials is nonzero. To prove the converse, we use the fact the ring of polynomials[30] is principal i.e. any ideal has a generator. The generator of the ideal generated by $U(z)$ and $V(z)$ is by definition their greatest common factor, so if they have no common factor we can find polynomials $a(z)$ and $b(z)$ such that $a(z)U(z) + b(z)V(z) = 1$. Thus, from $v(z)U(z) = u(z)V(z)$ we infer $v(z) = v(z)(a(z)U(z) + b(z)V(z)) = a(z)v(z)U(z) + b(z)v(z)V(z) = (a(z)u(z) + b(z)v(z))V(z)$ i.e. $v(z)$ must be a multiple of $V(z)$ which is impossible if $v(z)$ is not the zero polynomial but has degree $< n$.

Just as a side remark, the connection with elimination theory is the following. The equation $v(z)U(z) = u(z)V(z)$ where the degree of $u(z)$ is $< m$ and the degree of $v(z)$ is $< n$ can be expanded in powers of z, yielding a homogeneous linear system of size $(n+m) \times (n+m)$ for the $m+n$ unknown coefficients of $u(z)$ and $-v(z)$. It has a nontrivial solution if and only if the determinant

$$\begin{vmatrix}
U_0 & U_1 & \cdot & \cdot & U_m & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & U_0 & \cdot & \cdot & U_{m-1} & U_m & 0 & \cdot & \cdot & \cdot & 0 \\
\vdots & & & & & & & & & & \\
0 & 0 & \cdot & \cdot & \cdot & 0 & U_0 & U_1 & \cdot & \cdot & U_m \\
V_0 & V_1 & \cdot & \cdot & \cdot & V_n & 0 & \cdot & \cdot & \cdot & 0 \\
0 & V_0 & \cdot & \cdot & \cdot & V_{n-1} & V_n & 0 & \cdot & \cdot & 0 \\
\vdots & & & & & & & & & & \\
0 & 0 & \cdot & \cdot & 0 & V_0 & V_1 & \cdot & \cdot & \cdot & V_n
\end{vmatrix}$$

vanishes.

Coming back to the main course of the discussion, our aim is to show that because $Q(z)$ and $R(z)$ have no common factor, there is a unique polynomial $p(z)$ with degree strictly less that the degree of $R(z)$ such that $P(z) - p(z)Q(z)$ is a multiple of $R(z)$. The proof goes as follows. Given any polynomial $h(z)$, we can divide $P(z) - h(z)Q(z)$ by $R(z)$. We call the rest $q(z)$, a polynomial of degree strictly less that the degree of $R(z)$. The relation between $h(z)$ and $q(z)$ is affine. Thus restricting $p(z)$ to have degree strictly less that the degree of $R(z)$, we obtain an affine map between two affine spaces of the same dimension. Proposition 16 shows that the associated linear system is non singular: a non-trivial kernel would imply a common factor between $Q(z)$ and $R(z)$. Thus there is a single $p(z)$ of degree strictly less that the degree of $R(z)$ such that the division of $P(z) - p(z)Q(z)$ by $R(z)$ leaves no rest.

[30] Say over a field.

Then $\frac{P(z)}{Q(z)} - p(z)$ has no poles inside the contour C, i.e.

$$I := \oint_C \frac{dz}{R(z)} \frac{P(z)}{Q(z)} = \oint_C \frac{dz}{R(z)} p(z) = \lim_{z\to\infty} \frac{zp(z)}{R(z)}.$$

The last equality is because the contour can be deformed around ∞. Thus I is the ratio of the dominant coefficient of $p(z)$ by the dominant coefficient of $R(z)$.

As $p(z)$ is obtained by solving a linear system whose coefficients are certain coefficients of $P(z)$, $Q(z)$ and $R(z)$, it is clear that I is a rational function of those coefficients. It may look like a bad idea to compute $p(z)$ in full if only its dominant term is needed, but this is the most efficient way we have come by to compute such integrals.

3.3.2. Application to invariants

Remember that we need to compute

$$\oint_{|z|=1} \frac{dz}{z}(1-z^{-2}) \frac{1}{(1-tz^3)(1-tz)(1-tz^{-1})(1-tz^{-3})},$$

which we rewrite as

$$\oint_{|z|=1} dz \frac{z^3 - z}{(z-t)(z^3-t)(1-tz^3)(1-tz)}$$

It is easiest to deform and split the contour in two, one encircling the pole at $z = t$ and the other one encircling the 3 poles $z^3 = t$. The first residue is simply $\frac{1}{(1-t^2)(1-t^4)}$. To compute the contribution of the poles $z^3 = t$, we take $P(z) = z^3 - z$, $Q(z) = (z-t)(1-tz^3)(1-tz)$ and $R(z) = z^3 - t$. We look for a polynomial $p(z)$ of degree 2 such that $z^3 - z - p(z)(z-t)(1-tz^3)(1-tz)$ is divisible by $z^3 - t$. Solving the resulting linear system leads to

$$p(z) = -\frac{t^2z^2 + tz + 1}{(1-t^2)(1-t^4)}.$$

The z^2 coefficient is $-\frac{t^2}{(1-t^2)(1-t^4)}$ so that, taking the sum of all residues,

$$\oint_{|z|=1} \frac{dz}{z}(1-z^{-2}) \frac{1}{(1-tz^3)(1-tz)(1-tz^{-1})(1-tz^{-3})}$$

$$= \frac{1}{(1-t^2)(1-t^4)} - \frac{t^2}{(1-t^2)(1-t^4)} = \frac{1}{1-t^4}$$

as announced.

For the action on polynomials of degree 3, the weights are z^3, z, z^{-1}, z^{-3} as seen above. For polynomials of degree k they are $z^k, z^{k-2}, \cdots z^{-k+2}, z^{-k}$. Set $Q_k := \prod_{j=0}^{k}(1 - tz^{k-2j})$. By the same method as above, one computes explicitly the corresponding Molien series

$$F_k = \oint_{|z|=1} \frac{dz}{z}(1-z^{-2}) \frac{1}{Q_k(t,z)},$$

for the first values of k. One finds

$$F_0 = \frac{1}{1-t},\ F_1 = 1,\ F_2 = \frac{1}{1-t^2},\ F_3 = \frac{1}{1-t^4},\ F_4 = \frac{1}{(1-t^2)(1-t^3)},$$
$$F_5 = \frac{1+t^{18}}{(1-t^4)(1-t^8)(1-t^{12})},\ F_6 = \frac{1+t^{15}}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})},$$
$$F_7 = \frac{1+2t^8+4t^{12}+4t^{14}+5t^{16}+9t^{18}+6t^{20}+9t^{22}+8t^{24}+\cdots+t^{48}}{(1-t^4)(1-t^8)(1-t^{12})^2(1-t^{20})}$$

where for F_7 the coefficients are symmetric around the exponent 24 (for instance the coefficient of t^{24+8} is the same as the coefficient of t^{24-8}, i.e. 5).

The formulae for F_0 to F_6 can be used to give a clear picture of the ring of invariants, consistent with the general results quoted above. For instance the ring of invariants for homogeneous polynomials of degree 4 in two variables is a homogeneous polynomial algebra in 2 generators of respective degree 2 and 3. As another example, the ring of invariants for homogeneous polynomials of degree 6 in two variables is a module of dimension 2 over a polynomial algebra in 4 generators I_2, I_4, I_6, I_{10} (subscripts indicate the degrees), with a basis consisting of 1 and an invariant I_{15}, and I_{15}^2 must be a polynomial in I_2, I_4, I_6, I_{10}.

The example of invariants for homogeneous polynomials of degree 7 in two variables should serve as a warning. The Molien series suggest that it is a free 88 dimensional module over a polynomial algebra in 5 generators $I_4, I_8, I_{12}, I'_{12}, I_{20}$, with a basis consisting of 1, 2 invariants of degree 8, 4 invariants of degree 12, etc. The change in complexity from $k = 7$ to $k = 8$ is somewhat frightening.

We shall see later that the structure of invariants for homogeneous polynomials of degree 3 in 4 variables has a simple structure, but this appears to the author as a happy coincidence.

Let us conclude with some remarks.

(i) General theorems on invariants guarantee that F_k is a rational function of t.

(ii) For $k = 3, \cdots, 7$ the F_ks satisfy

$$F_k(t) = \frac{(-1)^k}{t^{k+1}} F_k(1/t).$$

This comes from the easy identity

$$Q_k(1/t, z) = (-t)^{-k-1} Q_k(1/t, z).$$

One should only be careful that the formula

$$F_k = \oint_{|z|=1} \frac{dz}{z}(1-z^{-2})\frac{1}{Q_k(t,z)},$$

defines F_k for small t. If t is moved around in the complex plane, the integration contour has to be deformed accordingly to perform the analytic continuation and the general formula is

$$F_k(t) - F_k^{(0)}(t) + F_k^{(\infty)}(t) = \frac{(-1)^k}{t^{k+1}} F_k(1/t),$$

where $F_k^{(0,\infty)}$ denote the residue of $\frac{dz}{z}(1-z^{-2})\frac{1}{Q_k(t,z)}$ at $0, \infty$. For $k \geq 3$ both residues vanish. It is puzzling to the author that the cases $k = 0, 1, 2$ involving correction terms occur precisely when the generic orbits do not have dimension $3 = \dim SL_2$, which is the explanation why for $k = 0, 1, 2$ the order of the pole at 1 behaves strangely, whereas it is $(k+1) - 3 = \dim \Sigma_{2,k} - \dim SL_2$ for $k \geq 3$.

(iii) Standard contour deformation arguments guarantee that F_k is singular exactly when a pole inside and a pole outside the integration contour pinch the contour.[31] By inspection, this means that F_k can only be singular when the polynomial $Q_k(t,z)$ acquires a double zero due to the fact that $z^l = t$ and $z^m = t$ have a common solution for some l, m equal to k mod 2 and such that $-k \leq l < 0 < m \leq k$. Then there is a singularity when $t^n = 1$ where $n = \frac{m-l}{\gcd(l,m)}$. This is enough to predict the full denominators of F_k, $k = 1, 2, \cdots, 5$. But things can be a bit more involved. For instance, we find that F_6 should simplify because the irreducible denominator can only be singular at $t^2 = 1$, $t^3 = 1$, $t^4 = 1$ and $t^5 = 1$. And indeed, the numerator $1 + t^{15}$ in the formula we gave for of F_6 contains a factor $1 + t^5$ that cancels the apparent singularity at $t^{10} = 1$ but $t^5 \neq 1$ suggested by the denominator. Of course, simplification by $1 + t^5$ spoils positivity of the coefficients in the numerator, which is not acceptable in the counting interpretation. In the same way, our formula for F_7 cannot be in reduced form because only $t^{10} = 1$, not $t^{20} = 1$ leads to a pinching singularity unavoidable by contour deformations.

To summarize, it is quite easy to identify all the poles of F_k and get a rough idea about the denominator and the degree of the numerator of F_k when written as a counting function (i.e. possibly not in reduced form), but the details of the numerator are much more intricate.

3.4. Cubics in $\mathbb{P}_3$: SL_4

Our aim is to study at last the action of SL_4 on polynomials $P_{4,3}$ homogeneous of degree 3 in four variables X_1, X_2, X_3, X_4. We write a homogeneous polynomial of degree k as

$$P_{4,k} = \sum_{k_1+k_2+k_3+k_4=k} c_{k_1,k_2,k_3,k_4} X_1^{k_1} X_2^{k_2} X_3^{k_3} X_4^{k_4}$$

3.4.1. Main results

The result on invariants is:

Theorem 4. *The ring of polynomial invariants is generated by five algebraically independent invariants of degrees* 8, 16, 24, 32, 40 *and an invariant of degree* 100 *whose square is a polynomial in the independent invariants.*

These invariants were produced in the middle of the 19th century [19], but a full proof completeness was only given more recently [1].

3.4.2. Brute force search for invariants

As the discussion of the simple case of binary cubics should amply have convinced the reader, the study of invariants of quaternary cubics by brute force is hopeless.

3.4.3. An invariant related to the existence of Eckhard points

In the case of cubics in $\mathbb{P}_1$, we observed that degeneracy of two points was obtained by imposing a single condition, which had to be an invariant, on the polynomial: this single condition was the vanishing of the (square of) the discriminant.

[31] By the way, this is used to locate singularities of Feynman integrals, another domain familiar to Raymond.

In Proposition 8 we exhibited an analogous phenomenon for Eckhard points, points that lie at the intersection of three lines on a cubic: generic cubics do not have Eckhard points, but we gave a deformation argument explaining that Eckhard points survive infinitesimal deformations satisfying a single condition. From their very definition, Eckhard points are covariant under linear transformations. Thus we may expect that the constraint for a cubic surface to admit an Eckhard point is given by a single invariant condition, i.e. by the vanishing of an invariant polynomial in the coefficients of the cubic. This is indeed true, and it turns out, see [5,4], that this invariant is the invariant of degree 100. Thus we see how intimate the relationship between invariant polynomials and purely geometric features of individual objects can be.

3.4.4. Canonical representatives

We quote the following theorem of Sylvester. It was the crucial ingredient used by Salmon [19,20] to give an explicit description of invariants.

Theorem 5 *(Sylvester). A general cubic surface can be transformed to the form*

$$\sum_0^4 a_i x_i^3 = 0 \qquad \sum_0^4 x_i = 0$$

with a_i, $i = 0, \cdots, 4$ determined up to permutation and scaling.

Here general means, as usual, that the exceptions depend on < 20 parameters, 20 being the number of parameters of a homogeneous cubic polynomial in 4 variables.

The theorem expresses a general cubic surface as the intersection of a cubic threefold with a hyperplane in $\mathbb{P}_4$, but the Sylvester form can be rewritten trivially as

$$a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + a_4 x_4^3 = a_0 (x_1 + x_2 + x_3 + x_4)^3$$

which expresses the canonical form as a cubic surface in $\mathbb{P}_4$ as it should be, at the expense of a slightly less symmetric expression.

The Sylvester form does not select a single point on each general orbit:

Proposition 17. *The subgroup of SL_4 fixing the Sylvester form is the pentahedral group order of* 480.

We shall not prove this fully. We shall exhibit 480 symmetry transformations, but leave completeness aside. The actions of permutations of $\{0, 1, 2, 3, 4\}$ on x_0, x_1, x_2, x_3, x_4 are obvious candidates, but they may fail to belong to SL_4. However, if σ is a permutation of $\{1, 2, 3, 4\}$ with signature ϵ and η is such that $\eta^4 = \epsilon$ then the substitution $x_i \to \eta x_{\sigma^{-1}(i)}$ belongs to SL_4. It permutes a_1, a_2, a_3, a_4 according to σ, and multiplies a_0, a_1, a_2, a_3, a_4 by η^{-3}. The substitution $x_1 \to \eta x_1, x_2 \to \eta x_2, x_3 \to \eta x_3, x_4 \to -\eta(x_1 + x_2 + x_3 + x_4)$ with $\eta^4 = -1$ transposes a_4 and a_0 and multiplies a_0, a_1, a_2, a_3, a_4 by η^{-3}. Thus each permutation of a_1, a_2, a_3, a_4 can be realized with 4 different multipliers, the multiplier η being such that $\eta^4 = \epsilon$ for a permutation with signature ϵ. This leads to a group of symmetry of the Sylvester form of order $480 = 5! \times 4$, a semi direct product of the permutation group on 5 letters by the cyclic group of order 4. This last action explains why the degrees of invariants must be multiples of 4.

3.4.5. Characters and Molien series

Again, the starting point is the diagonalization of the action on $\Sigma_{4,3}$ of diagonal matrices in SL_4. Write a general diagonal matrix in SL_4 as $d = diag(z_1, z_2/z_1, z_3/z_2, 1/z_3)$. Acting with d on a polynomial $P_{4,k}$ leads to the transformation rule

$$c'_{k_1,k_2,k_3,k_4} = c_{k_1,k_2,k_3,k_4} z_1^{k_2-k_1} z_2^{k_3-k_2} z_3^{k_4-k_3}$$

Thus $\Sigma_{4,3}$ split as a direct sum of 20 eigenspaces for the action of diagonal matrices, the eigenvalues for d being, starting from $(k_1, k_2, k_3, k_4) = (3, 0, 0, 0)$ and using lexicographic ordering,

$$\begin{gathered} z_1^{-3},\ z_1^{-1}z_2^{-1},\ z_1^{-2}z_2z_3^{-1},\ z_1^{-2}z_3,\ z_1z_2^{-2},\ z_3^{-1},\ z_2^{-1}z_3,\ z_1^{-1}z_2^2z_3^{-2},\ z_1^{-1}z_2,\ z_1^{-1}z_3^2, \\ z_1^3z_2^{-3},\ z_1^2z_2^{-1}z_3^{-1},\ z_1^2z_2^{-2}z_3,\ z_1z_2z_3^{-2},\ z_1,\ z_1z_2^{-1}z_3^2,\ z_2^3z_3^{-3},\ z_2^2z_3^{-1},\ z_2z_3,\ z_3^3. \end{gathered}$$

Setting $V = \mathbb{C}^4$ with the action of $G = SL_4$, we have $\Sigma_{4,3} = S_3(V^*)$. Note that this would give another route to get the above list of eigenvalues. The sum of these eigenvalues, which is the trace of the action of d on $\Sigma_{4,3}$, is denoted by $\chi(\Sigma_{4,3})$. The algebra of polynomials on $\Sigma_{4,3}$ is $S(S_3(V^*)^*)$ and by the general identity recalled in Appendix A we can write mechanically the character for this space, i.e. the generating function for the eigenvalues of d acting on $S(S_3(V^*)^*)$:

$$\sum_{k\geq 0} \chi(S_k(S_3(V^*)^*))t^k = \frac{1}{D(t, z_1, z_2, z_3)},$$

where

$$\begin{aligned} D(t, z_1, z_2, z_3) := {} & \left(1 - tz_1^{-3}\right)\left(1 - tz_1^{-1}z_2^{-1}\right)\left(1 - tz_1^{-2}z_2z_3^{-1}\right)\left(1 - tz_1^{-2}z_3\right) \\ & \left(1 - tz_1z_2^{-2}\right)\left(1 - tz_3^{-1}\right)\left(1 - tz_2^{-1}z_3\right)\left(1 - tz_1^{-1}z_2^2z_3^{-2}\right) \\ & \left(1 - tz_1^{-1}z_2\right)\left(1 - tz_1^{-1}z_3^2\right)\left(1 - tz_1^3z_2^{-3}\right)\left(1 - tz_1^2z_2^{-1}z_3^{-1}\right) \\ & \left(1 - tz_1^2z_2^{-2}z_3\right)\left(1 - tz_1z_2z_3^{-2}\right)(1 - tz_1)\left(1 - tz_1z_2^{-1}z_3^2\right) \\ & \left(1 - tz_2^3z_3^{-3}\right)\left(1 - tz_2^2z_3^{-1}\right)(1 - tz_2z_3)\left(1 - tz_3^3\right). \end{aligned}$$

We are looking for information about $S(S_3(V^*)^*)^G$. And as in the SL_2 case, there is a way to extract $\dim S_k(S_3(V^*)^*)^G$ for each k from the generating function above by an appropriate contour integral against a suitable measure. There are again a number of possibilities, all relying on the Weyl character formula. We simply quote the result, the interested reader is referred to Appendix B, in particular Sections B.5 and B.6, for motivation and detailed computations. Let $N(z_1, z_2, z_3)$ be the Laurent polynomial

$$\begin{aligned} N(z_1, z_2, z_3) := {} & \left(1 - \frac{z_1^2}{z_2}\right)\left(1 - \frac{z_1 z_2}{z_3}\right)\left(1 - \frac{z_2^2}{z_1 z_3}\right) \\ & (1 - z_1 z_3)\left(1 - \frac{z_2 z_3}{z_1}\right)\left(1 - \frac{z_3^2}{z_2}\right). \end{aligned}$$

The following holds:

Proposition 18. *Let W be a finite dimensional representation of $G = SL_4$. The action of diagonal matrices in SL_4 on W can be diagonalized. If $\chi(W) := \mathrm{Tr}_W\, d$ where $d := diag(z_1, z_2/z_1, z_3/z_2, 1/z_3)$ then $\chi(W)$ is a Laurent polynomial in z_1, z_2, z_3 and $\dim W^G$ is the constant term in $N(z_1, z_2, z_3)\chi(W)$, which can be collected by the contour integral*

$$\dim W^G = \oint \frac{dz_1}{z_1}\frac{dz_2}{z_2}\frac{dz_3}{z_3} N(z_1, z_2, z_3)\chi(W).$$

We infer that

$$\sum_{k\geq 0} \dim S_k(S_3(V^*)^*)^G t^k = \oint \frac{dz_1}{z_1}\frac{dz_2}{z_2}\frac{dz_3}{z_3}\frac{N(z_1, z_2, z_3)}{D(t, z_1, z_2, z_3)},$$

where the integration is a formal recipe to collect constant terms if the right-hand side is expanded in powers as a formal power series of t. It is easy to see, taking t as a complex parameter, that the small t expansion of $D(t, z_1, z_2, z_3)^{-1}$ converges if z_1, z_2, z_3 remain bounded away from 0 and ∞. Thus if the integration "contour" Γ respects this condition, for instance if it is taken as $\Gamma := \{|z_1| = |z_2| = |z_3| = 1\}$, the result will have the left-hand side as its small t expansion.

What remains to be done is a patient use of the method explained in Section 3.3:

(i) Start with the z_1 integration, say.
(ii) The variables t, z_2, z_3 being fixed, split $\frac{N(z_1,z_2,z_3)}{z_1 D(t,z_1,z_2,z_3)}$, seen as a function of z_1, as $\frac{1}{R(z_1)}\frac{P(z_1)}{Q(z_1)}$, where $\frac{1}{R(z_1)}$ contains the piece involving the poles at $|z_1| < 1$ assuming t is small.
(iii) Try to apply the procedure of Section 3.3.
(iv) If the procedure leads to a linear system that even your computer finds too complicated, try to split $R(z_1) = R_1(z_1)R_2(z_1)$ and deform the contour in two pieces, one encircling the zeros of $R_1(z_1)$ and the other the zeros of $R_2(z_1)$. Then apply the procedure of Section 3.3 twice, once to $\frac{1}{R_1(z_1)}\frac{P(z_1)}{Q_1(z_1)}$ where $Q_1(z_1) := Q(z_1)R_2(z_1)$ and once to $\frac{1}{R_2(z_1)}\frac{P(z_1)}{Q_2(z_1)}$ where $Q_1(z_1) := Q(z_1)R_2(z_1)$.
(v) If necessary split again until all smaller integrals can be computed and take the sum of all contributions to get a closed form for

$$\oint \frac{dz_1}{z_1}\frac{N(z_1, z_2, z_3)}{D(t, z_1, z_2, z_3)}$$

as a rational function of t, z_2, z_3.
(vi) Repeat the procedure by integrating z_2 say, and then z_3 to reach the final answer.

Unless some shortcuts unknown to the author do exist, carrying the program by hand seems difficult. Despite the fact that the final answer is relatively simple, intermediate computation show no obvious simplifications. For instance, the completion of the z_1 integral leads to a rational fraction it t, z_2, z_3. Even when reduced in lowest terms and after discarding some trivial factors, the numerator of this fraction involves 4354 distinct monomials (!), each with a sign and a non-trivial not so small coefficient.[32] Of course, it could be that this formidable expression is in fact the sum of a few simple fractions, but at least the fractions found when integrating along

[32] On the other hand, the numerator factorizes nicely and the poles can be predicted efficiently as explained in the simple case of SL_2 invariants in Section 3.3.2.

the natural smaller pieces of contour are not illuminating. Worse, they contain spurious poles that compensate in the sum but would lead to further complications to carry the next integrations if retained individually.

Even with the computer, things are not so direct. It turns out that intermediate computations can take a large time and/or a large amount of memory (dozens of gigabytes are easy to fill in with formal algebraic systems on such problems). Thus the above computation requires some patience and care. It seems that no one wrote it down before, and this can possibly be considered as the only modestly original piece of this contribution. The procedure of Section 3.3 is most likely known, but does not appear to our knowledge in the standard references on the computation of generating functions for invariants. It is crucial for the whole computation to work fast (now a few minutes on a laptop). Other methods we tried would exhibit an explosion of time and/or memory in the early steps, and a complete failure at some point.

The final answer is: for $|t|$ small (in fact for $|t| < 1$)

$$\oint_{|z_1|=|z_2|=|z_3|=1} \frac{dz_1}{z_1}\frac{dz_2}{z_2}\frac{dz_3}{z_3}\frac{N(z_1,z_2,z_3)}{D(t,z_1,z_2,z_3)} = \frac{1+t^{100}}{(1-t^8)(1-t^{16})(1-t^{24})(1-t^{32})(1-t^{40})}$$

which without surprise is totally consistent with the explicit computation of invariants carried by the fathers of group theory: the simplest way to generate the invariant ring is by choosing five independent invariants of degree 8, 16, 24, 32, 40 and an invariant of degree 100 whose square is a polynomial in the independent invariants.

As already explained, the pole of order 5 at $t = 1$ counts the dimension of the space of orbits: the total space of homogeneous cubic polynomials depends on 20 parameters, and a generic orbit has dimension 15 (the dimension of SL_4), leading to $20 - 15 = 5$ for the dimension of the space of orbits.

Notice that, just like for SL_2 invariants, there is a symmetry relating small and large t, not only in the final result, but also in the original integral representation: changing $t \to 1/t$ and interchanging z_1 and z_3 multiplies (formally) the integral by $-t^{-20}$. It can be shown that no accident happens at 0 and ∞ (see the discussion at the end of Section 3.3.2 for SL_2) so the symmetry is not only formal. The decrease as $-t^{-20}$ at large t plus the knowledge to the denominator $(1-t^8)(1-t^{16})(1-t^{24})(1-t^{32})(1-t^{40})$ would be enough to predict that the numerator has degree 100 and the dominant coefficient is 1. So once the expected five independent invariants are identified, the existence of an unexpected one comes along.

4. Conclusions

Our journey into the fascinating world of cubic surfaces is coming to an end. This was the occasion to meet many different mathematical characters (without pun), from elementary algebraic geometry to residue calculus, via elimination theory, group theory, combinatorics, etc. I realize how much Raymond has been present to my mind while writing these notes, and at the very moment of closing this tribute, I miss him bitterly. It is clear that Raymond should be credited for most of this work, and I would happily include him as an author, were it not for his high writing standards: these notes will remain a modest contribution to his memory.

Raymond played, and still plays, a very important role in my life. Knowing my tendency to lose things, I'm really happy that I kept all his letters over twenty five years carefully. I'm also very happy that my sun Thomas had the opportunity to meet Raymond. They really fitted well together, one of the reasons being their common passion for history. Thomas admired Raymond a lot and was always very eager to meet him on the occasion of our yearly visits to CERN. In our world dominated by money, violence, ostensibility and prejudice, Raymond embodied simplicity, tolerance, honesty and rigor.

Raymond maintained during his full lifetime an amazing passion for science, dedication to science and ability for science. Our last meeting took place in a room of the Hospital in Saint-Julien en Genevois on July 14, 2015. The first hour or so we discussed science, in particular projective invariants of cubic surfaces, a topic on which Raymond had recently made some nice, and possibly new, observations. Raymond looked tired, but he was as intellectually as sharp as ever. I remember that closing my eyes, and listening to his enthusiastic voice, I could forget about the time that had passed since our first meeting and imagine the conversation going on forever. Our last word on science was a quotation of Skorokhod's lemma [33] and its relation to quantum mechanics. Then we talked a little bit about the situation in France: politics, history, ... Raymond asked about Thomas and his studies. Up to that point, the conversation had taken a typical tour.[34] Then we talked about life in general for a while, a subject which was quite unusual between us. I remember him making the statement that "Life is very interesting in many respects, and I have had my share; at my age, any new day is a bonus to be enjoyed". I was so happy to be with him, but at the same time, I felt guilty that I was probably exhausting him. I did not know when to leave and how. But at some point Marie-Françoise entered the room, greeted me and kissed him tenderly: Raymond was in good hands and I left. I do not remember whether I knew, consciously or not, that I would not see him again.

The last letter I received from Raymond is dated May 22, 2015. As always, the handwriting was neat and characteristic. The conclusion was in typical Raymond style:

À suivre donc. [Compter les invariants] C'est comme la chasse aux papillons, il ne faut pas en laisser échapper.
Salut et fraternité[35].
RS

It took me a few more months to finish the counting. Now that it is automatized to the extent that I've been able to, it takes a few minutes and a few gigabytes of memory on a desk computer.

Appendix A. Reminder on algebra and combinatorics

Let V be a finite dimensional vector space of dimension $n \geq 1$ (over $\mathbb{C}$, say).

We start with a reminder (mostly for notation) on duality. The dual V^* of V is the space of linear forms on V, i.e. the set of linear maps from V to $\mathbb{C}$. For $v \in V$ and $\mu \in V^*$ we let $\langle \mu, v \rangle$ denote the pairing between V and V^*.

[33] Which states, informally, that any continuous function can be written, essentially in a unique way, as the difference of two continuous functions, a first one that is non-negative and a second one that is non-decreasing but remains constant on intervals when the first does not vanish.

[34] A standard phone call with Raymond was about $3/4$ of science, $1/5$ of general considerations on what had happened in the previous days, news of the family, and things like that. Only a very limited amount of time, if any, was given to his health problems.

[35] I learned of the special meaning of this greeting from Jean-Bernard Zuber.

A linear map l from V to W (another finite dimensional vector space over $\mathbb{C}$) induces a linear map ${}^t l$ (called the transpose of l) from W^* to V^* via $\langle {}^t l(\omega), v\rangle := \langle \omega, l(v)\rangle$ for $\omega \in W^*$ and $v \in V$.

We recall now a few elementary facts about tensor, symmetric and anti-symmetric algebras over a finite dimensional vector space. We introduce a basis, but the reader should keep in mind that intrinsic definitions can be given,[36] see e.g. [16].

We recall that the tensor algebra $T(V)$ over V is a graded associative algebra with unit. As a vector space, it is the direct sum:

$$T(V) := \sum_{k\geq 0} T_k(V).$$

The standard notation for $T_k(V)$ is $V^{\otimes k}$. Fix a basis $(v_1, \cdots, v_n)$ of V. Then $T_k(V)$ has a basis made of symbols v_I where $I \in \{1, \cdots, n\}^k$. The unit element is $v_\emptyset \in T_0(V) \equiv \mathbb{C}$, and for $k = 1$ $v_{\{m\}} \equiv v_m$, $m \in \{1, \cdots, n\}$. The product in the basis is given by concatenation, i.e. $v_I v_J = v_{IJ}$, $IJ = (i_1, \cdots, i_k, j_1, \cdots, j_l)$ if $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_l)$ and then extended to all of $T(V)$ by multi-linearity. So $v_I = v_{i_1} \cdots v_{i_k}$ for $I = (i_1, \cdots, i_k)$. Clearly $\dim T_k(V) = n^k$.

Starting from $T(V)$ one defines the symmetric algebra $S(V)$ and the antisymmetric algebra $A(V)$ by taking quotients. For $S(V)$ (resp. $A(V)$) one imposes the (resp. anti-)commutation relations $v_i v_j = v_j v_i$ (resp. $v_i v_j = -v_j v_i$) for $i, j \in \{1, \cdots, n\}$. These are homogeneous relations so $S(V)$ and $A(V)$ are graded algebras, $S(V)$ is commutative and $A(v)$ is graded-commutative. Then $S_k(V)$ (resp. $A_k(V)$) has a basis made of symbols (the equivalence classes of) v_I where $I = (i_1, \cdots, i_k)$ is such that $i_1 \leq i_2 \cdots \leq i_k$ (resp. $i_1 < i_2 \cdots < i_k$).

When a group G acts on V, it also acts on V^* and $S(V^*)$, which is by definition the set of polynomial functions on V. One of our main interest in what follows is the study of $S(V^*)^G$, the set (in fact an algebra) of polynomials on V invariant under G.

Writing

$$S(V) := \sum_{k\geq 0} S_k(V), \;\; A(V) := \sum_{k\geq 0} A_k(V)$$

one finds[37] $\dim S_k(V) = \frac{(n+k-1)!}{(n-1)!k!}$ for $k \geq 0$ (resp. $\dim A_k(V) = \frac{n!}{(n-k)!k!}$ for $0 \leq k \leq n$ and $\dim A_k(V) = 0$ for $k > n$).

Our aim is to refine this dimension counting.

Observe that $T_1(V)$, $S_1(V)$ and $A_1(V)$ are all isomorphic to V. If Λ is a linear operator on V, we can thus define a unique linear operator $T(\Lambda)$ on $T(V)$ in such a way that $\Lambda(v_I v_J) = \Lambda(v_I)\Lambda(v_J)$ for each tuples $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_l)$. Note that $T(\Lambda)$ maps each $T_k(V)$ into itself. As $\Lambda(v_i v_j \pm v_j v_i) := \Lambda(v_i)\Lambda(v_j) \pm \Lambda(v_j)\Lambda(v_i)$ and the right-hand side is a linear combinations of the $v_{i'} v_{j'} \pm v_{j'} v_{i'}$, the extension of Λ to $T(V)$ descends to $S(V)$ (resp. $A(V)$) where we denote it by $S(\Lambda)$ (resp. $A(\Lambda)$). Again, $S(\Lambda)$ (resp. $A(\Lambda)$) maps each $S_k(V)$ (resp. $A_k(V)$) into itself.

Note that these extensions satisfy $T(\Lambda\Lambda') = T(\Lambda)T(\Lambda')$ (resp. $S(\Lambda\Lambda') = S(\Lambda)S(\Lambda')$, $A(\Lambda\Lambda') = A(\Lambda)A(\Lambda')$). This is the crucial property that is exploited to define a representation of a group G on $T(V)$ (resp. $S(V)$ and $A_(V)$) from a representation of G on V, see Subsection 3.1.

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[36] Alternatively, one could check that the definitions given here do in fact not depend on the basis.

[37] Is is not fortuitous that this is exactly the dimension of the space of homogeneous polynomials in n variables of degree k which we computed in Proposition 2.

Now suppose that Λ is diagonal in the basis $(v_1, \cdots, v_n)$ with $\Lambda(v_i) := \lambda_i v_i$. Then $T(\Lambda)$, $S(\Lambda)$ and $A(\Lambda)$ are also diagonal in the corresponding basis we defined above.

Then $\operatorname{Tr}_V \Lambda = \sum_{i=1}^n \lambda_i$. We may view the λ_i as complex numbers or more generally as formal weights i.e. independent commuting variables. Our aim is to compute $\operatorname{Tr}_{T_k(V)} T(\Lambda)$, $\operatorname{Tr}_{S_k(V)} S(\Lambda)$ and $\operatorname{Tr}_{A_k(V)} A(\Lambda)$.

Proposition 19. *If t is a variable, the formal power series $\operatorname{Tr}_{T(V)} T(t\Lambda)$, $\operatorname{Tr}_{S(V)} S(t\Lambda)$ and $\operatorname{Tr}_{A(V)} A(t\Lambda)$ make sense and*

$$\begin{aligned}
\operatorname{Tr}_{T(V)} T(t\Lambda) &= \frac{1}{1 - t \operatorname{Tr}_V \Lambda} \\
\operatorname{Tr}_{S(V)} S(t\Lambda) &= \frac{1}{\operatorname{Det}_V (1 - t\Lambda)} \\
\operatorname{Tr}_{A(V)} A(t\Lambda) &= \operatorname{Det}_V (1 + t\Lambda).
\end{aligned}$$

The first relation is obvious, it just says that $\operatorname{Tr}_{T_k(V)} T(\Lambda) = (\operatorname{Tr}_V \Lambda)^k$. The second one is obtained by rearrangement of a formal power series. By definition

$$\begin{aligned}
\operatorname{Tr}_{S(V)} S(t\Lambda) &= \sum_{k \geq 0} \operatorname{Tr}_{S_k(V)} S(t\Lambda) = \sum_{k \geq 0} t^k \operatorname{Tr}_{S_k(V)} S(\Lambda) \\
&= \sum_{k \geq 0} t^k \sum_{i_1 \leq \cdots \leq i_k} \lambda_{i_1} \cdots \lambda_{i_k}.
\end{aligned}$$

Now, each $\lambda_{i_1} \cdots \lambda_{i_k}$ $i_1 \leq \cdots \leq i_k$ can be written uniquely as $\lambda_1^{k_1} \cdots \lambda_n^{k_n}$ for an unique n-tuple $k_1, \cdots, k_n \geq 0$ such that $k_1 + \cdots + k_n = k$, and vice versa. Thus

$$\sum_{k \geq 0} \sum_{i_1 \leq \cdots \leq i_k} t^k \lambda_{i_1} \cdots \lambda_{i_k} = \sum_{k_1, \cdots k_n \geq 0} t^{k_1 + \cdot k_n} \lambda_1^{k_1} \cdots \lambda_n^{k_n} = \prod_{m=1}^{n} \left(\sum_{k_m \geq 0} t^{k_m} \lambda_m^{k_m} \right).$$

This leads to

$$\operatorname{Tr}_{S(V)} S(t\Lambda) = \prod_{m=1}^{n} \frac{1}{1 - t\lambda_m} = \frac{1}{\operatorname{Det}_V (1 - t\Lambda)}.$$

The antisymmetric case goes along the same lines except that $0 \leq k_1, \cdots, k_n \leq 1$.

Note that the identity $\operatorname{Tr}_{S(V)} S(t\Lambda) \operatorname{Tr}_{A(V)} A(-t\Lambda) = 1$ is one of the basic boson–fermion supersymmetric identities.

Appendix B. Reminder on roots, weights and characters

We start by recalling how the complex Lie algebra $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C})$ of $SL_n := SL_n(\mathbb{C})$ fits in the general theory of (semi-)simple finite dimensional Lie algebras (see e.g. the remarkable [8]). We shall not try to address the question of the relation between irreducible finite dimensional representations of the Lie group SL_n and the Lie algebra $\mathfrak{sl}_n$, except to quote that any finite dimensional representation of $\mathfrak{sl}_n$ is automatically a representation of SL_n, and the converse is true, assuming some smoothness. Irreducibility is preserved in both directions.

B.1. Basic definitions

The Lie algebra $\mathfrak{sl}_n$ of SL_n consists of traceless matrices, as can be seen informally by imposing the condition $\operatorname{Det} M = 1$ to an $n \times n$ matrix of the form $M = \mathrm{Id} + \varepsilon N$ to first order in ε, leading to $\operatorname{Tr} N = 0$.

For $i, j \in [1, n]$ we define e_{ij} as the matrix all of whose entries are 0 except for a 1 at the intersection of line i and column j. Note that $e_{ij} e_{kl} = \delta_{jk} e_{il}$. Thus

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}.$$

The e_{ij} form a linear basis of the vector space $M_n(\mathbb{C})$ of all $n \times n$ matrices. We let $\mathfrak{d}$ denote the subspace of diagonal matrices in $M_n(\mathbb{C})$.

Note that e_{ij} is traceless, i.e. belongs to $\mathfrak{sl}_n$, if and only if $i \neq j$.

One can split $\mathfrak{sl}_n$ as

$$\mathfrak{sl}_n = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$$

where $\mathfrak{g}_-$ is the space of lower triangular matrices, $\mathfrak{g}_0 \subset \mathfrak{d}$ the space of diagonal traceless matrices and $\mathfrak{g}_+$ the space of upper triangular matrices. Just as $\mathfrak{sl}_n$, these three vector spaces are stable under commutation, i.e. are Lie algebras on their own. Indeed, the $e_{i,j}$s, $j < i$ (resp. $j > i$) form a basis of $\mathfrak{g}_-$ (resp. $\mathfrak{g}_+$) and the above commutation relations show the stability.

B.2. Roots

Let $i \neq j$. From

$$[e_{kk}, e_{ij}] = (\delta_{ki} - \delta_{jk}) e_{ij}$$

we infer that if $d \in \mathfrak{d}$ then $[d, e_{ij}] = \alpha_{ij}(d) e_{ij}$ for some non-zero linear form $\alpha_{ij} \in \mathfrak{d}^*$ and $\alpha_{ij} = -\alpha_{ji}$.

As $\mathfrak{g}_0 \subset \mathfrak{d}$, $\mathfrak{g}_0^*$ is a natural quotient of $\mathfrak{d}^*$, i.e. any linear form on $\mathfrak{d}$ induces a linear form on $\mathfrak{g}_0$. To keep the distinction visible, we use the bracket notation for the duality pairing between $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$, but keep the same notation for an element of $\mathfrak{d}^*$ and its image in $\mathfrak{g}_0^*$. Thus, if $h \in \mathfrak{g}_0$ and $i \neq j$, then $[h, e_{ij}] = \langle \alpha_{ij}, h \rangle e_{ij}$. In particular, the action of $\mathfrak{g}_0$ on $\mathfrak{g}_\pm$ is diagonalizable. It is clear that $\mathfrak{g}_0$ is abelian, and a simple computation shows that the space of matrices in $\mathfrak{sl}_n$ commuting with $\mathfrak{g}_0$ is $\mathfrak{g}_0$ itself, so $\mathfrak{g}_0$ is a maximal abelian sub-algebra[38] of $\mathfrak{sl}_n$.

The linear forms α_{ij}, $i \neq j$, on $\mathfrak{g}_0$ are called the roots of $\mathfrak{sl}_n$. A root is called positive (resp. negative) if $j > i$ (resp. $j < i$). We let R (resp R^+, R^-) denote the set of roots (resp. positive, negative roots). If $\alpha \in R$, say $\alpha = \alpha_{ij}$ for some $i \neq j$, we set $\mathfrak{g}_\alpha := \mathbb{C} e_{ij}$, so that if $x \in \mathfrak{g}_\alpha$ and $h \in \mathfrak{g}_0$, we have $[h, x] = \langle \alpha, h \rangle x$ and

$$\mathfrak{sl}_n = \mathfrak{g}_0 \oplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Note that each $\mathfrak{g}_\alpha$ is one-dimensional.

Either by explicit computation or by invocation of the Jacobi identity, one checks that for $\alpha, \beta \in R$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. For $\alpha + \beta = 0$, we interpret the subscript 0 in $\mathfrak{g}_0$ as the zero element in $\mathfrak{g}_0^*$. Note however that 0 is not counted as a root, even though $[\mathfrak{g}_0, \mathfrak{g}_0] = 0$.

[38] Also called a Cartan sub-algebra once the fact that $\mathfrak{sl}_n$ is semi-simple is established.

Clearly the $h_i := e_{ii} - e_{i+1,i+1}$ for $i = 1, \cdots, n-1$ form a basis of $\mathfrak{g}_0$. Set $\alpha_i := \alpha_{i,i+1}$ for $i = 1, \cdots, n-1$. From $[h_i, e_{jk}] = (\delta_{ij} - \delta_{ik} - \delta_{i+1,j} + \delta_{i+1,k})e_{jk}$ we infer that for $j > k$ we have $\alpha_{jk} = \alpha_j + \cdots + \alpha_{k-1}$ and $C_{ij} := \langle \alpha_i, h_j \rangle = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j}$ is a positive symmetric matrix.[39] Thus the α_i form a basis of $\mathfrak{g}_0^*$ and every positive root is a linear combination of α_is with non-negative integer coefficients. Due to this property, the α_is are called the simple positive roots. Thus the linear integral combinations of the (simple) roots form a lattice Λ_R, of maximal rank in $\mathfrak{g}_0^*$, called the root lattice.

B.3. Representations and weights

For $j > i$ we have

$$[e_{ij}, e_{ji}] = e_{ii} - e_{jj} \quad [e_{ii} - e_{jj}, e_{ij}] = e_{ij} \quad [e_{ii} - e_{jj}, e_{ji}] = -e_{ji}.$$

Thus setting, for $j \neq i$, $E_\alpha := e_{ij}$ and $H_\alpha := e_{ii} - e_{jj}$ we infer that, for $\alpha \in R^+$ $E_{+\alpha}$, $E_{-\alpha}$ and H_α generate a copy $\mathfrak{s}_\alpha$ of $\mathfrak{sl}_2$, in that their commutation relations are a copy of those of E_+, E_- and H as given in Section 3.2.2:

$$[E_{+\alpha}, E_{-\alpha}] = H_\alpha \qquad [H_\alpha, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}.$$

These commutation relations tell us that $\langle \alpha, H_\alpha \rangle = 2$, and that H_{α_i} is what we called h_i before.

A representation of $\mathfrak{sl}_n$ is automatically a representation of $\mathfrak{s}_\alpha$ for each positive root α. From the general structure of finite-dimensional representations of $\mathfrak{sl}_2$ one concludes that, for each root α, H_α can be diagonalized in any finite-dimensional representation of $\mathfrak{sl}_n$. But the H_αs commute among themselves, so they can even be diagonalized simultaneously. Thus, as the H_αs span $\mathfrak{g}_0$, if V is a finite-dimensional representation of $\mathfrak{sl}_n$ there is a finite collection of linear forms $\mu \in \mathfrak{g}_0^*$ such that

$$V = \oplus_\mu V_\mu \text{ with } V_\mu \neq \{0\} \text{ and } h.v = \langle \mu, h \rangle v \text{ for } h \in \mathfrak{g}_0 \text{ and } v \in V_\mu.$$

The linear forms μ are called the weights of V and the V_μ the weight spaces. Moreover, $\mathfrak{g}_\alpha V_\mu \subset V_{\mu+\alpha}$ for each root and each weight space (with the obvious convention that $V_\nu = \{0\}$ if ν is not a weight of V).

If for V we take $\mathfrak{sl}_n$ itself with action $x.v := [x, v]$ (the adjoint action, a representation by the Jacobi identity) we find that the roots are themselves weights, and $\mathfrak{sl}_n = \mathfrak{g}_0 \oplus_{\alpha \in R} \mathfrak{g}_\alpha$ is the weight-space decomposition of $\mathfrak{sl}_n$.

A further consequence of the general $\mathfrak{sl}_2$ theory is that the eigenvalues of H_α on any finite-dimensional representation V of $\mathfrak{sl}_n$ must be integers. We conclude that $\mu(H_\alpha) \in \mathbb{Z}$ for each weight μ and each root α. These conditions define a lattice Λ_W called the weight lattice. It has maximal rank in $\mathfrak{g}_0^*$ because it contains the root lattice Λ_R.

This quantization condition has an important consequence. We've been silent on why finite dimensional representations of $\mathfrak{sl}_n$ are automatically representations of SL_n. We can at least see that the diagonal matrices of SL_n act on any finite dimensional representation of $\mathfrak{sl}_n$. Indeed, a diagonal matrix Λ in SL_n is of the form $diag(\lambda_1, \cdots, \lambda_n)$ where $\lambda_1, \cdots, \lambda_n \in \mathbb{C}^*$ and $\lambda_1 \cdots \lambda_n = 1$. Hence, using the fact the exponential function maps $\mathbb{C}$ onto $\mathbb{C}^*$, Λ can be written as

[39] The matrix C_{ij} is easy to diagonalize by Fourier transform: the eigenvalues are $\lambda^{(m)} := 4\sin^2 \frac{\pi m}{2n}$, $m = 1, \cdots, n-1$ with corresponding eigenvector $v_j^{(m)} = \sqrt{\frac{2}{n}} \sin \frac{\pi m j}{n}$.

$$\Lambda = e^{\sum_{j=1}^{n-1} t_j H_{\alpha_j}} = diag(e^{t_1}, e^{t_2 - t_1}, \cdots, e^{t_{n-1} - t_{n-2}}, e^{-t_{n-1}}),$$

for suitably chosen $t_1, \cdots, t_n \in \mathbb{C}$. But the t_j's are not completely determined by Λ, and the ambiguity is that each t_j is defined modulo addition of an arbitrary integral multiple of $2i\pi$. But if $V = \oplus_\mu V_\mu$ is the weight space decomposition of a finite dimensional representations of $\mathfrak{sl}_n$, H_{α_j} acts on V_μ by multiplication by $\langle \mu, H_{\alpha_j} \rangle$ so letting Λ act on V_μ by multiplication by

$$\begin{aligned} &e^{\sum_{j=1}^{n-1} t_j \mu(H_{\alpha_j})} \\ &= \lambda_1^{\langle \mu, H_{\alpha_1} + \cdots + H_{\alpha_{n-1}} \rangle} \lambda_2^{\langle \mu, H_{\alpha_2} + \cdots + H_{\alpha_{n-1}} \rangle} \cdots \lambda_{n-2}^{\langle \mu, H_{\alpha_{n-2}} + H_{\alpha_{n-1}} \rangle} \lambda_{n-1}^{\langle \mu, H_{\alpha_{n-1}} \rangle} \end{aligned}$$

is unambiguous precisely because each $\langle \mu, H_{\alpha_j} \rangle$ belongs to $\mathbb{Z}$.

We check that for $i \neq j$ and $k \neq l$ we have $\langle \alpha_{kl}, H_{\alpha_{ij}} \rangle = \delta_{ik} - \delta_{il} - \delta_{jk} + \delta_{jl}$. In particular we recover $\alpha(H_\alpha) = 2$ for each $\alpha \in R$, as already noticed above as a consequence of $[H_\alpha, E_\alpha] = 2E_\alpha$.

We define the set of so-called fundamental weights μ_i, $i = 1, \cdots, n-1$ by

$$\langle \mu_i, H_{\alpha_j} \rangle = \delta_{ij}.$$

The fundamental weights form a basis of the weight lattice Λ_W. Indeed, we know that, for $j > i$, $H_{\alpha_{ij}} = e_{ii} - e_{jj} = \sum_{i \leq k < j} H_{\alpha_k}$ so the above conditions on the μ_is are necessary and sufficient conditions for their integral linear combinations to take integral values on each H_α.

As $C_{ij} = \langle \alpha_i, H_{\alpha_j} \rangle = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j}$ we infer immediately that $\alpha_i = 2\mu_i - \mu_{i+1} - \mu_{i-1}$ (with the obvious convention $\mu_0 = \mu_n = 0$). This relation with integral coefficients illustrates that $\Lambda_R \subset \Lambda_W$. The Cartan matrix C_{ij} can be inverted to express the fundamental weights in terms of the simple roots.

B.4. The Weyl group

Observe that $\mathfrak{g}_\alpha V_\mu \subset V_{\mu+\alpha}$ for $\alpha \in R$. Fixing $\alpha \in R$, acting repeatedly on V_μ with $E_{\pm\alpha}$ builds a representation V' of $\mathfrak{s}_\alpha$, which decomposes as $V' = \oplus_{k \in [k_-, k_+]} V_{\mu + k\alpha}$ for an appropriate interval $[k_-, k_+]$ of integers. Note that this has to be a direct sum because H_α acts on (non-trivial) $V_{\mu+k\alpha}$ by as the scalar $\mu(H_\alpha) + k\alpha(H_\alpha)$ and $\alpha(H_\alpha) \neq 0$. By the general theory for $\mathfrak{sl}_2$, the eigenspaces in V' with opposite eigenvalues of the action of H_α have the same dimension: for some integer k we have $\mu(H_\alpha) + k\alpha(H_\alpha) = -\mu(H_\alpha)$ and defining σ_α on $\nu \in \mathfrak{g}_0^*$ by

$$\sigma_\alpha(\nu) := \nu - \frac{2\langle \nu, H_\alpha \rangle}{\langle \alpha, H_\alpha \rangle} \alpha = \nu - \langle \nu, H_\alpha \rangle \alpha$$

we have that $\sigma_\alpha(\mu)$ is a weight whenever μ is one, and that $\dim V_{\sigma_\alpha(\mu)} = \dim V_\mu$.

The transformations σ_α are involutive and they generate a group W of transformations of $\mathfrak{g}_0^*$ called the Weyl group of $\mathfrak{sl}_n$. And we have shown that the Weyl group permutes weight spaces in every finite dimensional representation of $\mathfrak{sl}_n$. The transformations σ_α are in fact reflexions: for each root α the condition $\langle \nu, H_\alpha \rangle = 0$ defines a hyperplane in $\mathfrak{g}_0^*$, σ_α acts as 1 on this hyperplane and as -1 on $\mathbb{C}\alpha$. Thus the determinant of σ_α is -1, and the determinant $\epsilon(w)$ of each $w \in W$ is ± 1.

In the case $n = 2$ we find that $\Lambda_R = 2\mathbb{Z}$, $\Lambda_W = \mathbb{Z}$, and $\sigma(\mu) = -\mu$ for $\mu \in \mathbb{Z}$, thereby recovering that the Weyl group has order 2 and the non-trivial element changes H into $-H$.

Observe that, for $h \in \mathfrak{g}_0$ one has $\langle \sigma_\alpha(\nu), h\rangle = \langle \nu, h - \langle \alpha, h\rangle H_\alpha\rangle$, so that the Weyl group W acts on $h \in \mathfrak{g}_0$ by

$$\sigma_\alpha(h) = h - \langle \alpha, h\rangle H_\alpha.$$

We have used the fact that σ_α is an involution, standard duality would define the right-hand side as the action of σ_α^{-1}.

To identify W, we argue as follows. First the equation for the action of σ_α on $\mathfrak{g}_0$ extends to an action on $\mathfrak{d}$: if $d \in \mathfrak{d}$, define[40]

$$\sigma_\alpha(d) = d - \alpha(d) H_\alpha.$$

Using $H_{\alpha_{ij}} = e_{ii} - e_{jj}$, the action of $\sigma_{\alpha_{ij}}$ on e_{kk} is seen to be

$$\sigma_{\alpha_{ij}}(e_{kk}) = \begin{cases} e_{kk} & \text{if } k \neq i, j \\ e_{jj} & \text{if } k = i \\ e_{ii} & \text{if } k = j \end{cases}$$

Thus if τ_{ij} acts on $[1, n]$ as $\tau_{ij}(k) = \begin{cases} k & \text{if } k \neq i, j \\ j & \text{if } k = i \\ i & \text{if } k = j \end{cases}$, i.e. if τ_{ij} is the transposition of i and k within the symmetric group $\mathfrak{S}_n$ acting on n letters, we see that $\sigma_{\alpha_{ij}}(e_{kk}) = e_{\tau_{ij}(k), \tau_{ij}(k)}$. As is well-known, transpositions generate the full symmetric group. Thus W acting on $\mathfrak{d}$ is isomorphic to $\mathfrak{S}_n$, and $\mathfrak{d}$ can be seen as the fundamental representation of $\mathfrak{S}_n$ permuting the basis vectors e_{kk}, $k \in [1, n]$. The vector $\sum_k e_{kk}$ is invariant, and the complementary subspace $\mathfrak{g}_0$ is (by construction) globally invariant. Thus the action of W on $\mathfrak{g}_0$ exhibits it as isomorphic to $\mathfrak{S}_n$.

Armed with is result, we can prove the following:

Proposition 20. *The images of the open cone*

$$\mathring{C} := \{\mu \in \mathfrak{g}_0^*, \langle \mu, H_{\alpha_i}\rangle > 0 \, \textit{for } i = 1, \cdots, n-1\}$$

under the action of the Weyl group are disjoint. Moreover,

$$\{\mu \in \mathfrak{g}_0^*, \langle \mu, H_\alpha\rangle \neq 0 \, \textit{for all roots } \alpha\} = \cup_{w \in W} w(\mathring{C}) \textit{ is a partition.}$$

To prove the first part, we observe that $\mathring{C} = \{\mu \in \mathfrak{g}_0^*, \langle \mu, H_\alpha\rangle > 0 \; \forall \, \alpha \in R^+\}$ because all positive roots are positive (integral) linear combinations of the simple roots, and then $\langle \mu, H_\alpha\rangle < 0$ for $\alpha \in R^-$. Thus, if $\mu \in \mathring{C}$ and $\sigma \in W$ is such that $\sigma(\mu) \in \mathring{C}$ we have $\langle \sigma(\mu), \alpha_{ij}\rangle > 0$ when $j > i$, i.e. $\langle \mu, \alpha_{\sigma^{-1}(i), \sigma^{-1}(j)}\rangle > 0$ when $j > i$. This holds if and only if $\alpha_{\sigma^{-1}(i), \sigma^{-1}(j)}$ is a positive root (i.e. if $\sigma^{-1}(j) > \sigma^{-1}(i)$) when $j > i$. That means, σ preserves the ordering of $[1, n]$ and must by the identity in $\mathfrak{S}_n$. Thus, if $\mu \in \mathring{C}$, all the iterates of μ are distinct. We may even go a bit further: if $\langle \mu, H_\alpha\rangle \neq 0$ for all roots α, define $j \tilde{>} i$ if $\langle \mu, H_{\alpha_{ij}}\rangle > 0$. We claim that $\tilde{>}$ is a strict ordering of $[1, n]$. Indeed, for $i \neq j$, either $j \tilde{>} i$ or $i \tilde{>} j$ but not both, and if $k \tilde{>} j$ and $j \tilde{>} i$, then $i \neq k$ and using $H_{\alpha_{ik}} = H_{\alpha_{ik}} + H_{\alpha_{ik}}$ we get $\langle \mu, H_{\alpha_{ik}}\rangle > 0$ i.e. $k \tilde{>} i$. Then there is a single permutation transforming the relation $>$ in $\tilde{>}$, so we have proved the second part of the proposition.

[40] Recall that for $i \neq j$ we have defined the linear forms α_{ij} on $\mathfrak{d}$ by $\alpha_{ij}(e_{kk}) = \delta_{ki} - \delta_{jk}$.

In particular, the permutation reversing the order, i.e. for whom $\tilde{>}$ is exactly $<$ is the only one sending R^+ to R^- and it induces the transformation $\mu \to -\mu$ on $\mathfrak{g}_0^*$.

The action of W on $\mathfrak{g}_0$ can be obtained by automorphisms. The normalizer N of $\mathfrak{g}_0$, i.e. the subgroup of SL_n consisting of the Ks such that $K\mathfrak{g}_0 K^{-1} = \mathfrak{g}_0$, acts linearly on $\mathfrak{g}_0$. The centralizer C of $\mathfrak{g}_0$, i.e. the subgroup of $\mathfrak{sl}_n$ consisting of the Ks such that $KhK^{-1} = h$ for every $h \in \mathfrak{g}_0$, is a normal subgroup of N that acts trivially, and $W = N/C$.

The proof is elementary. First we identify N. If $K \in N$ and $v \in \mathfrak{g}_0$ are given, there is $v' \in \mathfrak{g}_0$ such that $Kh = h'K$ i.e. in components $K_{ij}v_j = v'_i K_{ij}$. As K is invertible, for each i there are js such that $K_{ij} \neq 0$, and for those js, $v_j = v'_i$. Choosing a v all of whose components are distinct, we see that for each i there is in fact exactly one j such that $K_{ij} \neq 0$, so there is a permutation σ of $[1, n]$ and non-zero coefficients k_j such that $K_{ij} = \delta_{i,\sigma(j)} k_j$. Then $K = PD$ where P is the permutation matrix $P_{ij} = \delta_{i,\sigma(j)}$ and D is the diagonal matrix $D_{ij} = \delta_{ij} k_j$. The group N consists of matrices of that form with Det $D =$ Det P. Then $h \to KhK^{-1}$ acts on $\mathfrak{g}_0$ just like the permutation associated to K. An analogous but simpler argument shows that C is the group of diagonal matrices of determinant 1. The fact that $W = N/C$ follows.

B.5. Classification of finite dimensional irreducible representations, characters

A representation of V of $\mathfrak{sl}_n$ is reducible if it contains a non-trivial (i.e. different from $\{0\}$ and V itself) sub-representation, and irreducible otherwise.

We let Λ_W^+ be the integral cone generated by the fundamental weights, i.e. $\Lambda_W^+ := \{\sum_{i=1}^{n-1} m_i \mu_i, m_i \in \mathbb{N}\}$.

The basic results are:

Theorem 6. *The finite dimensional representation V of $\mathfrak{sl}_n$ are completely reducible, i.e. they split as direct sums of irreducible representations.*

and

Theorem 7. *Every irreducible finite dimensional representation V of $\mathfrak{sl}_n$ contains a single weight space annihilated by $\mathfrak{g}_+$. The weight μ is called the highest of V, μ belongs to Λ_W^+ and the corresponding weight space is 1-dimensional. Conversely, for each $\mu \in \Lambda_W^+$, there is (modulo equivalence) an unique irreducible representation space V with highest weight μ.*

The result is checked by explicit examination for $\mathfrak{sl}_2$ but the general proof for $\mathfrak{sl}_n$, a fortiori for all semi-simple Lie algebras, requires some serious work (see e.g. [8,16,15,9,10,18]). If V is an arbitrary representation of $\mathfrak{sl}_n$ and μ is a highest weight, we denote by mult$(R^{(\mu)}, V)$ the number of summands with highest weight μ in the splitting of V as a direct sum of irreducibles.

Once the classification theorem is known, one would like to describe the irreducible representation V with highest weight μ more precisely. In particular, if $V = \oplus_\nu V_\nu$ is the weight space decomposition of V, what is the dimension of V_ν? We also need tools to compute efficiently mult$(R^{(\mu)}, V)$ when V is an arbitrary finite dimensional representations of $\mathfrak{sl}_n$. For instance if V and V' are irreducible representations with highest weight μ and μ', we would like to compute the multiplicities mult$(R^{(\mu'')}, V \otimes V')$ of the tensor product representation $V \otimes V'$. The basic tool to answer such questions is remarkable. It can be given several forms. One of them, that we shall not quote, is the so-called Freudenthal formula. Another one is given in terms of generating functions. We take some notation. Let $z_1, \cdots, z_{n-1}$ be a set of indeterminates. If $\mu = \sum_{i=1}^{n-1} m_i \mu_i$ is any element of Λ_W, we set

$$z^\mu := z_1^{m_1} \cdots z_{n-1}^{m_{n-1}},$$

a Laurent monomial. Also, we set $\rho = \sum_{i=1}^{n-1} \mu_i$, the sum of the fundamental weights.

Definition 5. Let V be a representation of $\mathfrak{sl}_n$ and let $V = \oplus_\nu V_\nu$ be its weight space decomposition. The character $\chi(V)$ of V is the Laurent polynomial

$$\chi(V) := \sum_\nu \dim V_\nu z^\nu.$$

The z_i, $i = 1, \cdots, n-1$ above are formal variables, but any diagonal matrix $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$ in SL_n can be written as $\Lambda = (z_1, z_2/z_1, \cdots, z_{n-1}/z_{n-2}, 1/z_{n-1})$ for uniquely defined non-zero complex numbers $z_1, \cdots, z_{n-1}$: $z_1 = \lambda_1, z_2 = \lambda_1\lambda_2, \cdots, z_{n-1} = \lambda_1 \cdots \lambda_{n-1}$. We know that Λ acts diagonally on each weight space, and z^ν is nothing but the eigenvalue of Λ acting on V_ν. Thus if the formal variables z_is are traded for complex numbers and $\Lambda = diag(z_1, z_2/z_1, \cdots, z_{n-1}/z_{n_2}, 1/z_{n-1})$, we have

$$\chi(V) = \mathrm{Tr}_V \Lambda$$

This is a more standard interpretation of characters. We have seen in Appendix A how characters behave under algebraic constructions like tensors products for instance.

The following is crucial:

Theorem 8 *(The Weyl character formula). Let V be the irreducible representation of $\mathfrak{sl}_n$ with highest weight $\mu \in \Lambda_W^+$. Then*

$$\chi(V) = \frac{\sum_{w \in W} \epsilon(w) z^{w(\rho+\mu)}}{\sum_{w \in W} \epsilon(w) z^{w(\rho)}}.$$

This formula makes sense because Laurent polynomials in $z_1, \cdots, z_{n-1}$ form an algebra without zero divisors. For later use, we denote by Δ the numerator in the character formula.

We also quote:

Theorem 9 *(The Weyl denominator formula). The following identity holds:*

$$\Delta := \sum_{w \in W} \epsilon(w) z^{w(\rho)} = z^\rho \prod_{\alpha \in R^+} (1 - z^{-\alpha}).$$

For $\mathfrak{sl}_n$, we have seen that there is a transformation $w \in W$ such that $w(\mu) = -\mu$ on $\mathfrak{g}_0$, and that it corresponds to the order reversing permutation of $[1, n]$. From this and the denominator formula, one can deduce with a modest amount of explicit computation that $2\rho = \sum_{\alpha \in R^+} \alpha$.

B.6. Applications of the Weyl character formula

Let V be a representation of $\mathfrak{sl}_n$. The following is just a rewriting of the fact that the character of a direct sum is the sum of characters:

Proposition 21.

$$\chi(V) = \sum_{\mu\in\Lambda_W^+} \text{mult}(R^{(\mu)}, V)\frac{\sum_{w\in W}\epsilon(w)z^{w(\rho+\mu)}}{\sum_{w\in W}\epsilon(w)z^{w(\rho)}}$$

A simple but useful consequence is

Proposition 22. *For any $w \in W$ and $\mu \in \Lambda_W^+$, the multiplicity* $\text{mult}(R^{(\mu)}, V)$ *is $\epsilon(w)$ times the coefficient of the monomial $z^{w(\rho+\mu)}$ in the expansion of $\Delta\chi(V)$ as a Laurent polynomial.*

To prove this, we write the previous formula as

$$\Delta\chi(V) = \sum_{w\in W}\sum_{\mu\in\Lambda_W^+} \epsilon(w)z^{w(\rho+\mu)}\,\text{mult}(R^{(\mu)}, V)$$

If $\mu \in \Lambda_W^+$ then $\rho+\mu$ belongs to the open cone $\mathring{C} := \{\mu \in \mathfrak{g}_0^*, \langle\mu, H_{\alpha_i}\rangle > 0 \text{ for } i = 1, \cdots, n-1\}$ so by Proposition 20 the images of $\rho+\mu$ under W are all distinct: the orbits have maximal size. Hence if $\mu, \mu' \in \Lambda_W^+$ are distinct, their orbits under W are disjoint. So all the monomials on the right-hand side are linearly independent, proving the announced result.

This leads to the following integral representation, where $\oint \frac{dz}{z}$ stands for $\oint \frac{dz_1}{z_1}\cdots\oint\frac{dz_{n-1}}{z_{n-1}}$ (remember each contour integral contains the canonical factor $(2i\pi)^{-1}$):

Proposition 23. *For any $w \in W$,*

$$\text{mult}(R^{(\mu)}, V) = \epsilon(w)\oint \frac{dz}{z} z^{-w(\rho+\mu)}\Delta\chi(V).$$

In particular, for $w = 1$,

$$\text{mult}(R^{(\mu)}, V) = \oint \frac{dz}{z} z^{-\mu}\prod_{\alpha\in R^+}(1-z^{-\alpha})\chi(V).$$

In particular, for $w = 1$ and $\mu = 0$, the highest weight of the trivial representation,

$$\dim V^G = \text{mult}(R^{(0)}, V) = \oint \frac{dz}{z}\prod_{\alpha\in R^+}(1-z^{-\alpha})\chi(V).$$

The second and third formulae make use of the Weyl denominator formula. The contour integral measure $\oint \frac{dz}{z}$ could be taken as nothing but a formal notation to select the constant term in the Laurent polynomial it acts on. But it can also be used by doing integral calculus with contours for $z_1, \cdots, z_{n-1}$ encircling 0 once in the positive direction but otherwise arbitrary. The first formula, holding for any $w \in W$, can be averaged over W to yield

Proposition 24.

$$\mathrm{mult}(R^{(\mu)}, V) = \frac{1}{|W|} \oint \frac{dz}{z} \left(\sum_{w \in W} \epsilon(w) z^{-w(\rho+\mu)} \right) \Delta \chi(V).$$

In particular,

$$\mathrm{mult}(R^{(\mu)}, V) = \frac{1}{|W|} \oint_{|z_1|=\cdots=|z_{n-1}|=1} \frac{dz}{z} |\Delta|^2 \chi(V) \overline{\chi(R^{(\mu)})}.$$

The second formula holds because on $|z_1| = \cdots = |z_{n-1}| = 1$ taking inverses is the same as taking complex conjugates, so that

$$\sum_{w \in W} \epsilon(w) z^{-w(\rho)} = \overline{\Delta}$$

and by the Weyl character formula

$$\sum_{w \in W} \epsilon(w) z^{-w(\rho+\mu)} = \overline{\Delta} \overline{\chi(R^{(\mu)})}$$

The second formula in Proposition 24 says that the characters of the different irreducible finite dimensional representations of SL_n form an orthonormal set for the measure $\frac{1}{|W|} \oint_{|z_1|=\cdots=|z_{n-1}|=1} \frac{dz}{z} |\Delta|^2$. The formula has its roots (no pun) in the representation theory of the compact subgroup SU_n of SL_n. Any unitary matrix can be diagonalized, i.e. the conjugacy class of any matrix in SU_n contains a diagonal matrix. If the eigenvalues are distinct, which is the generic situation, the diagonal form is unique up to permutation of the diagonal entries, i.e. up to the action of the Weyl group W. Defining a class function f on SU_n as a function invariant under conjugation $f(klk^{-1} = f(l))$ for $k, l \in SU_n$, we see that the trace $\mathrm{Tr}_V\, l$ in any representation, i.e. any character is a class function. One can also view class function as functions on diagonal unitary matrices invariant under permutation.

One of the beautiful consequences of the Peter–Weyl theorem is that the characters of irreducible unitary representations of SU_n (they are automatically finite dimensional and are given by restriction to SU_n of the finite dimensional irreducible representations of SL_n) form an orthonormal basis for the space of square integrable class functions on SU_n, the measure being precisely $\frac{1}{|W|} \oint_{|z_1|=\cdots=|z_{n-1}|=1} \frac{dz}{z} |\Delta|^2$.

From a practical viewpoint, Proposition 23 is much better than Proposition 24 to use on a computer, because one spares at least a factor $|W|$ (i.e. $n!$ for $\mathfrak{sl}_n$).

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